CHARACTERIZATIONS OF PRETAMENESS AND THE ORD-CC

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ABSTRACT. The notion of pretameness was introduced by Sy Friedman in order to characterize the preservation of the axioms of ZF without the power set axiom for class forcing over models of ZF. We present several new characterizations of pretameness, for instance in terms of the forcing theorem, the forcing equivalence of partial orders and their dense suborders and the existence of nice names for sets of ordinals. Furthermore, for most properties under consideration we also present a corresponding characterization of the Ord-chain condition.

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1. INTRODUCTION

This paper is motivated by the question which properties of set forcing carry over to the context of class forcing, in particular to pretame class forcing and to class forcing with the Ord-chain condition. We show that the properties mentioned in the abstract hold for all pretame notions of class forcing, and that they can in fact be used to characterize pretameness and the Ord-chain condition in various ways. A list of the main results of this paper can be found at the end of this section. In order to properly state them, we will need to introduce our setup together with some basic notation. This is mostly the same as in [HKL$^+$] and [HKS].

We will work with transitive second-order models of set theory, that is models of the form $\mathbb{M} = \langle M, \mathcal{C} \rangle$, where M is transitive and denotes the collection of *sets* of $\mathbb{M}$ and $\mathcal{C}$ denotes the collection of *classes* of $\mathbb{M}$.[1] We require that $M \subseteq \mathcal{C}$ and that elements of $\mathcal{C}$ are subsets of M, and we call elements of $\mathcal{C} \setminus M$ *proper classes* (of $\mathbb{M}$). Classical transitive first-order models of set theory are covered by our approach when we let $\mathcal{C}$ be the collection of classes definable over $\langle M, \in \rangle$. The theories that we will be working in will be fragments of *Kelley-Morse set theory* KM, and mostly we will work within fragments of *Gödel-Bernays set theory* GB. By GB^- we denote GB without the power set axiom (but with Collection rather than Replacement), and by GBC we denote GB together with the axiom of global choice.[2] By a countable transitive model of some second order theory, we mean a transitive second-order model $\mathbb{M} = \langle M, \mathcal{C} \rangle$ of such a theory with both M and $\mathcal{C}$ countable in V. Most of the time, we will avoid using the power set axiom, however we will often make use of the following consequence of GB:

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[1] Arguing in the ambient universe V, we will sometimes refer to classes of such a model $\mathbb{M}$ as sets, without meaning to indicate that they are sets of $\mathbb{M}$. In particular this will be the case when we talk about subsets of M.

[2] For more details, see [HKS] or [HKL$^+$]. For a detailed axiomatization of KM, see [Ant15].

Definition 1.1. A model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^- *has a hierarchy* if there is $C \in \mathcal{C}$ such that

(1) $C \subseteq \mathrm{Ord}^M \times M$;
(2) For each $\alpha \in \mathrm{Ord}^M$, $C_\alpha = \{x \in M \mid \exists \beta < \alpha\,[\langle \beta, x\rangle \in C]\} \in M$;
(3) If $\alpha < \beta$ in Ord^M then $C_\alpha \subseteq C_\beta$;
(4) $M = \bigcup_{\alpha \in \mathrm{Ord}^M} C_\alpha$.

If C defines a hierarchy on M, then the *C-rank* of $x \in M$, denoted $\mathsf{rnk}_C(x)$, is the least $\alpha \in \mathrm{Ord}^M$ such that $x \in C_{\alpha+1}$.

Remark 1.2. (1) Assume that $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$. Suppose that $\mathcal{C}$ contains a well-order $\prec$ of M which is *good*, i.e. for every $x \in M$, $C_x = \{y \in M \mid y \prec x\} \in M$. Then the order-type of $\langle M, \prec\rangle$ is Ord^M and $C = \{\langle x, y\rangle \mid x \in \mathrm{Ord}^M \wedge y \prec x\}$ witnesses that $\mathbb{M}$ has a hierarchy. Conversely, it is easy to check that every well-order of M in $\mathcal{C}$ that has order-type Ord^M is good.
(2) If $\mathbb{M} \models \mathsf{GB}^-$ has a hierarchy and $\mathcal{C}$ contains a well-order of M, then $\mathcal{C}$ contains a good well-order of M (see [HKS, Remark 1.3]).
(3) Without a hierarchy, the existence of a well-order of M in $\mathcal{C}$ does not imply that $\mathcal{C}$ contains a well-order of M of order-type Ord^M. This can be seen as follows. Let N be a countable transitive model of ZFC in which CH fails and such that there is a definable wellorder of $H(\omega_1)$; assuming the existence of a countable transitive model of ZFC, such a model exists by [Har77]. Let $M = H(\omega_1)^N$ and let $\mathcal{C}$ be the collection of subsets of M that are definable over M. It follows that $\langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ and that $\mathcal{C}$ contains a well-order of M. Since CH fails in N, $\mathcal{C}$ however cannot contain a well-order of M of order-type Ord^M.
(4) If $\mathbb{M} \models \mathsf{GB}^-$ has a hierarchy, then it satisfies *representatives choice* (see [HKL$^+$, Definition 3.2]). Assuming the presence of a hierarchy, we will thus sometimes cite results from [HKL$^+$] that assume representatives choice without further mention.

Fix a countable transitive model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^-. By a *notion of class forcing* (for $\mathbb{M}$) we mean a separative partial order $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ such that $P, \leq_{\mathbb{P}} \in \mathcal{C}$.[3] We will frequently identify $\mathbb{P}$ with its domain P. In the following, we also fix a notion of class forcing $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ for $\mathbb{M}$. Note that the assumption of separativity does not restrict us much, since if $\mathbb{M}$ satisfies representatives choice, one can always pass from a notion of class forcing to its separative quotient (see [HKL$^+$]).

We call σ a *$\mathbb{P}$-name* if all elements of σ are of the form $\langle \tau, p\rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. Define $M^{\mathbb{P}}$ to be the set of all $\mathbb{P}$-names that are elements of M and define $\mathcal{C}^{\mathbb{P}}$ to be the set of all $\mathbb{P}$-names that are elements of $\mathcal{C}$. In the following, we will usually call the elements of $M^{\mathbb{P}}$ simply *$\mathbb{P}$-names* and we will call the elements of $\mathcal{C}^{\mathbb{P}}$ *class $\mathbb{P}$-names*. If $\sigma \in M^{\mathbb{P}}$ is a $\mathbb{P}$-name, we define

$$\operatorname{rank} \sigma = \sup\{\operatorname{rank} \tau + 1 \mid \exists p \in \mathbb{P}\ \langle \tau, p\rangle \in \sigma\}$$

to be its *name rank*. We will sometimes also need to use the usual set theoretic rank of some $\sigma \in M$, which we will denote as $\mathsf{rnk}(\sigma)$.

We say that a filter G on $\mathbb{P}$ is *$\mathbb{P}$-generic over* $\mathbb{M}$ if G meets every dense subset of $\mathbb{P}$ that is an element of $\mathcal{C}$. Given such a filter G and a $\mathbb{P}$-name σ, we recursively define the *G-evaluation* of σ as

$$\sigma^G = \{\tau^G \mid \exists p \in G\ \langle \tau, p\rangle \in \sigma\},$$

and similarly we define Γ^G for $\Gamma \in \mathcal{C}^{\mathbb{P}}$. Moreover, if G is $\mathbb{P}$-generic over $\mathbb{M}$, then we set $M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$ and $\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$. Furthermore, we call $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$ a *$\mathbb{P}$-generic extension* of $\mathbb{M}$. $\mathbb{M}[G]$ is what we call a *generic class extension* in [HKS, Section 2], where we argue that given a $\mathbb{P}$-generic filter G, $\mathbb{M}[G]$ is the canonical $\mathbb{P}$-generic extension of $\mathbb{M}$. In the present paper, this will be the only form of generic extension that we consider.

Given an $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_{m-1}, \vec{\Gamma})$, where $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ are class name parameters, $p \in \mathbb{P}$ and $\vec{\sigma} \in (M^{\mathbb{P}})^m$, we write $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\vec{\sigma}, \vec{\Gamma})$ if for every $\mathbb{P}$-generic filter G over $\mathbb{M}$ with $p \in G$, $\langle M[G], \Gamma_0^G, \ldots, \Gamma_{n-1}^G\rangle \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G)$.

A fundamental result in the context of set forcing is the *forcing theorem*. It consists of two parts, the first one of which, the so-called *definability lemma*, states that the forcing relation is definable

[3] Note that this differs from the definition of notions of class forcing in [HKL$^+$] and [HKS], where we do not make the assumption of separativity (and in [HKL$^+$], we also do not assume antisymmetry, i.e. we allow for the more general notion of a preorder).

in the ground model, and the second part, denoted as the *truth lemma*, says that every formula which is true in a generic extension $M[G]$ is forced by some condition in the generic filter G. In the context of second-order models of set theory, this has the following natural generalization:

Definition 1.3. Let $\varphi \equiv \varphi(v_0, \ldots, v_{m-1}, \vec{\Gamma})$ be an $\mathcal{L}_\in$-formula with class name parameters $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$.

(1) We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$ if
$$\{\langle p, \sigma_0, \ldots, \sigma_{m-1}\rangle \in P \times (M^{\mathbb{P}})^m \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \vec{\Gamma})\} \in \mathcal{C}.$$
(2) We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$ if for all $\sigma_0, \ldots, \sigma_{m-1} \in M^{\mathbb{P}}$, for all $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with
$$\langle M[G], \Gamma_0^G, \ldots, \Gamma_{n-1}^G\rangle \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G),$$
there is $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \vec{\Gamma})$.
(3) We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$ if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

Note that by [HKL$^+$, Theorem 1.3], the forcing theorem can fail in class forcing; in fact, even the truth lemma may fail for atomic formulae (see [HKL$^+$, Theorem 1.5]). There are, however, many known properties of class forcing notions which guarantee that the forcing theorem holds (see [Zar73], [Fri00], [HKL$^+$] and [HKS]).

We identify sequences of the form $\langle C_i \mid i \in I\rangle$ for classes $C_i \in \mathcal{C}$ and $I \in \mathcal{C}$ with their *code* $\{\langle c, i\rangle \mid c \in C_i \wedge i \in I\}$. In particular, we say such a sequence is an element of $\mathcal{C}$ if its code is.

Definition 1.4. [Fri00] A notion of forcing $\mathbb{P}$ for $\mathbb{M}$ is *pretame* for $\mathbb{M} \models \mathsf{GB}^-$ if for every $p \in \mathbb{P}$ and for every sequence of dense classes $\langle D_i \mid i \in I\rangle \in \mathcal{C}$ with $I \in M$, there is $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I\rangle \in M$ such that for every $i \in I$, $d_i \subseteq D_i$ and d_i is predense below q.

As shown by Sy Friedman in [Fri00], pretame notions of class forcing satisfy the forcing theorem over models of GB. For the benefit of the reader, we will provide the proof of this result in a generalized setting at the beginning of Section 2. Furthermore, we will show that in a certain sense, pretameness can in fact be characterized by the forcing theorem.

If $\mathbb{M} \models X \subseteq \mathsf{KM}$, we say that a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ *preserves* X if $\mathbb{M}[G] \models X$. The significance of pretameness lies in the observation that it characterizes the preservation of the axioms of GB^- over models of GB^- with a hierarchy. This topic shall be discussed in Section 3.

Definition 1.5. We say that a notion of class forcing $\mathbb{P}$ satisfies the Ord-*chain condition* (or simply Ord-*cc*) over $\mathbb{M}$ if every antichain of $\mathbb{P}$ which is in $\mathcal{C}$ is already in M.

Note that if $\mathcal{C}$ contains a good well-order of M, then the Ord-cc is strictly stronger than pretameness. On the other hand, in the absence of such a well-order it is not even clear whether every notion of class forcing with the Ord-cc satisfies the forcing theorem (see Question 7.2).

A property that is closely related to the forcing theorem and that can be used to characterize pretameness and the Ord-cc is the existence of a Boolean completion. We distinguish between two types of Boolean completions, depending on whether suprema exist for all sets or for all classes of conditions.

Definition 1.6. If $\mathbb{B}$ is a Boolean algebra, then $\mathbb{B}$ is

(1) M*-complete* if the supremum $\sup_{\mathbb{B}} A$ exists in $\mathbb{B}$ for every $A \in M$ with $A \subseteq \mathbb{B}$.
(2) $\mathcal{C}$*-complete* if the supremum $\sup_{\mathbb{B}} A$ exists in $\mathbb{B}$ for every $A \in \mathcal{C}$ with $A \subseteq \mathbb{B}$.

Definition 1.7. We say that $\mathbb{P}$ *has a Boolean* M*-completion in* $\mathbb{M}$ if there is an M-complete Boolean algebra $\mathbb{B} = \langle B, 0_{\mathbb{B}}, 1_{\mathbb{B}} \neg, \wedge, \vee\rangle$ such that B, all Boolean operations of $\mathbb{B}$ and an injective dense embedding from $\mathbb{P}$ into $\mathbb{B}\setminus\{0_{\mathbb{B}}\}$ are elements of $\mathcal{C}$. Similarly, we define a *Boolean* $\mathcal{C}$*-completion* of $\mathbb{P}$ to be a Boolean M-completion $\mathbb{B}$ of $\mathbb{P}$ which additionally is $\mathcal{C}$-complete.

In set forcing, Boolean completions are unique: If $\mathbb{B}_0$ and $\mathbb{B}_1$ are both Boolean completions of $\mathbb{P}$ and $e_0 : \mathbb{P} \to \mathbb{B}_0$ and $e_1 : \mathbb{P} \to \mathbb{B}_1$ are dense embeddings, then one can define an isomorphism from $\mathbb{B}_0$ to $\mathbb{B}_1$ by setting $f(b) = \sup\{e_1(p) \mid p \in \mathbb{P} \wedge e_0(p) \leq b\}$ for $b \in \mathbb{B}_0$. Moreover, f fixes $\mathbb{P}$ in the sense that $f(e_0(p)) = e_1(p)$ for every condition $p \in \mathbb{P}$. In class forcing, this proof works only for

Boolean $\mathcal{C}$-completions. It follows from results in [HKL$^+$, Section 9] that Boolean M-completions need not be *unique* in the following sense.

Definition 1.8. We say that a notion of class forcing $\mathbb{P}$ *has a unique Boolean M-completion in* $\mathbb{M}$, if $\mathbb{P}$ has a Boolean M-completion $\mathbb{B}_0$ in $\mathbb{M}$ and for every other Boolean M-completion $\mathbb{B}_1$ of $\mathbb{P}$ in $\mathbb{M}$ there is an isomorphism in V between $\mathbb{B}_0$ and $\mathbb{B}_1$ which fixes $\mathbb{P}$. The property that $\mathbb{P}$ *has a unique Boolean $\mathcal{C}$-completion* is defined correspondingly.

In section 4 we will investigate the relationship between the existence of Boolean completions on the one hand, and pretameness and the Ord-cc on the other.

A technique that we will use in many places throughout this paper is adding suprema to proper classes of conditions. Recall that for $A \in \mathcal{C}$ with $A \subseteq \mathbb{P}$, we say that $p \in \mathbb{P}$ is the *supremum* of A (denoted $p = \sup_{\mathbb{P}} A$) if and only if

(1) $\forall a \in A\ a \leq_{\mathbb{P}} p$ and
(2) A is predense below p.

We describe a general method of how to extend a notion of class forcing by adding suprema. Let $S = \langle X_i \mid i \in I\rangle \in \mathcal{C}$ with $I \in M$ be a sequence of subclasses of $\mathbb{P}$. Making use of a suitable bijection in $\mathcal{C}$, we may assume that $P \cap I = \emptyset$. Now let $\mathbb{P}_S$ be the forcing notion with domain $P_S = P \cup I$ ordered by

$$\begin{aligned}
i \leq_{\mathbb{P}_S} p &\iff \forall q \in X_i\ q \leq_{\mathbb{P}} p,\\
p \leq_{\mathbb{P}_S} i &\iff X_i \text{ is predense below } p \text{ in } \mathbb{P},\\
i \leq_{\mathbb{P}_S} j &\iff \forall q \in X_i\ q \leq_{\mathbb{P}_S} j.
\end{aligned}$$

For $i \in I$, we will usually write $\sup X_i$ rather than i. In case that $\sup X_i$ already exists in $\mathbb{P}$, or that $\sup X_i \leq_{\mathbb{P}_S} \sup X_j$ and $\sup X_j \leq_{\mathbb{P}_S} \sup X_i$ for some $i \neq j$, instead of $\mathbb{P}_S$ we need to consider the quotient of $\mathbb{P}_S$ by the equivalence relation $p \sim q$ iff $p \leq_{\mathbb{P}_S} q$ and $q \leq_{\mathbb{P}_S} p$ for $p, q \in P \cup I$, in order to obtain a separative partial order. Since $I \in M$ and $\mathbb{P}$ is separative, all equivalence classes are set-sized, this can easily be done and we will identify $\mathbb{P}_S$ with this quotient in this case. We call $\mathbb{P}_S$ the *forcing notion obtained from $\mathbb{P}$ by adding $\sup X_i$ for all $i \in I$*. Note that by construction, $\mathbb{P}$ is dense in $\mathbb{P}_S$.

Lemma 1.9. *Suppose that $\mathbb{P}$ is a notion of class forcing which satisfies the forcing theorem. If $S \in \mathcal{C}$ is a finite sequence of subclasses of $\mathbb{P}$, then $\mathbb{P}_S$ satisfies the forcing theorem.*

Proof. This is an easy generalization of [HKL$^+$, Lemma 9.3]. □

It will follow from the proof of Theorem 2.6 that this may fail if I is infinite.

Example 1.10. We will frequently use the following notion of class forcing introduced in [Fri00] to motivate our results. Given $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$, let $\mathrm{Col}(\omega, \mathrm{Ord})^M$ denote the notion of forcing with conditions of the form $p\colon \mathrm{dom}(p) \to \mathrm{Ord}^M$ for $\mathrm{dom}(p) \subseteq \omega$ finite, ordered by reverse inclusion. It follows from [HKL$^+$, Lemma 2.2, Lemma 6.3 and Theorem 6.4] that $\mathrm{Col}(\omega, \mathrm{Ord})^M$ satisfies the forcing theorem. However, $\mathrm{Col}(\omega, \mathrm{Ord})^M$ is non-pretame witnessed by the sequence $\langle D_n \mid n \in \omega\rangle$, where D_n is the dense class of all conditions p such that $n \in \mathrm{dom}(p)$. Another – simpler – way to see this is to observe that any $\mathrm{Col}(\omega, \mathrm{Ord})^M$-generic filter gives rise to a cofinal sequence from ω to Ord^M and thus induces a failure of Replacement in the generic extension (see Section 3).

Results.

Notation. Let $\mathbb{M} \models \mathsf{GB}^-$ and let Ψ be some property of a notion of class forcing $\mathbb{P}$ for $\mathbb{M} = \langle M, \mathcal{C}\rangle$. We say that $\mathbb{P}$ *densely* satisfies Ψ if every notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$, for which there is a dense embedding in $\mathcal{C}$ from $\mathbb{P}$ into $\mathbb{Q}$, satisfies the property Ψ.

For the sake of simplicity, if there is a dense embedding in $\mathcal{C}$ from $\mathbb{P}$ into $\mathbb{Q}$, we will frequently assume that $\mathbb{P}$ is a suborder of $\mathbb{Q}$. This does not constitute a restriction, since $\mathbb{P}$ is always isomorphic to a dense suborder of $\mathbb{Q}$.

The following two theorems summarize the results from the present paper. Regarding the results of Theorem 1.11 below, (1) is (in a slightly less general context) due to Sy Friedman ([Fri00]), the nontrivial direction of the equivalence of (1) and (2) was shown in [HKS, Theorem 5.1] and the equivalence of (4) and (5) was shown in [HKL$^+$].

Theorem 1.11. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M, and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. The following properties (over $\mathbb{M}$) are equivalent to the pretameness of $\mathbb{P}$ over $\mathbb{M}$, where we additionally require the non-existence of a first-order truth predicate for (4) and (5), and for (7) we assume that $\mathbb{M} \models \mathsf{KM}$, which is notably incompatible to the assumptions used for (4) and (5).*

(1) *$\mathbb{P}$ preserves GB^-/Collection/Replacement.*
(2) *$\mathbb{P}$ satisfies the forcing theorem and preserves Separation.*
(3) *$\mathbb{P}$ satisfies the forcing theorem and does not add a cofinal/surjective/bijective function from some $\gamma \in \mathrm{Ord}^M$ to Ord^M.*
(4) *$\mathbb{P}$ densely satisfies the forcing theorem.*
(5) *$\mathbb{P}$ densely has a Boolean M-completion.*
(6) *$\mathbb{P}$ satisfies the forcing theorem and produces the same generic extensions as $\mathbb{Q}$ for every forcing notion $\mathbb{Q}$ such that $\mathcal{C}$ contains a dense embedding from $\mathbb{P}$ into $\mathbb{Q}$.*[4]
(7) *$\mathbb{P}$ densely has the property that every set of ordinals in any of its generic extensions has a nice name.*

Given notions of class forcing $\mathbb{P}$ and $\mathbb{Q}$ for $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$, given $\pi \in \mathcal{C}$ such that π is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$, and given a $\mathbb{P}$-name σ, we recursively define $\pi(\sigma) = \{\langle \pi(\tau), \pi(p)\rangle \mid \langle \tau, p\rangle \in \sigma\}$.

Theorem 1.12. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M, and let $\mathbb{P}$ be a notion of class forcing that satisfies the forcing theorem. The following properties (over $\mathbb{M}$) are equivalent:*

(1) *$\mathbb{P}$ satisfies the* Ord*-cc.*
(2) *$\mathbb{P}$ satisfies the maximality principle.*[5]
(3) *$\mathbb{P}$ densely has a unique Boolean M-completion.*
(4) *$\mathbb{P}$ has a Boolean $\mathcal{C}$-completion.*
(5) *If there are $\mathbb{Q}, \pi \in \mathcal{C}$ such that π is a dense embedding from $\mathbb{P}$ to $\mathbb{Q}$ and $\sigma \in M^{\mathbb{Q}}$, then there is $\tau \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \pi(\tau)$.*
(6) *$\mathbb{P}$ densely has the property that whenever $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \subseteq \check{\alpha}$ for some $\sigma \in M^{\mathbb{P}}$ and $\alpha \in \mathrm{Ord}^M$ then there is a nice $\mathbb{P}$-name τ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma = \tau$.*

Whether pretameness implies (6) of Theorem 1.11 was a question (that turned out to have an easy positive answer) posed to us by Victoria Gitman at the European Set Theory Workshop in Cambridge in the summer of 2015, and was one of the starting points of the research that we present in this paper.

2. The forcing theorem

In this section, we characterize pretameness in terms of the forcing theorem. We will make use of the following theorem, which is an easy adaptation of [Fri00, Theorem 2.18] to our generalized setting. For the benefit of the reader, we nevertheless include its proof.

Theorem 2.1 (Sy Friedman)**.** *Let $\mathbb{M}$ be a model of GB^- with a hierarchy and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. If $\mathbb{P}$ is pretame over $\mathbb{M}$ then $\mathbb{P}$ satisfies the forcing theorem over $\mathbb{M}$.*

Proof. Suppose that $C = \langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ witnesses that $\mathbb{M}$ has a hierarchy. Observe first that by [HKL$^+$, Theorem 4.3] it suffices to check the definability of the forcing relation of $\mathbb{P}$ for atomic formulae. To achieve this, we construct a class function $F : \mathbb{P} \times M^{\mathbb{P}} \times M^{\mathbb{P}} \times 2 \to M \times 2$ in $\mathcal{C}$ such that for $p \in \mathbb{P}$ and $\sigma, \tau \in M^{\mathbb{P}}$, $F(p, \sigma, \tau, i) = \langle d, j\rangle$ for some nonempty set $d \subseteq \{q \in \mathbb{P} \mid q \leq_{\mathbb{P}} p\}$ and for all $q \in d$, q decides either $\sigma \in \tau$ (in case $i = 0$) or $\sigma = \tau$ (in case $i = 1$) either positively (if $j = 1$) or negatively (if $j = 0$).[6] Given such F, we can define the $\mathbb{P}$-forcing relation by

$$p \Vdash_{\mathbb{P}} \sigma \in \tau \iff \forall q \leq_{\mathbb{P}} p\, \exists d\ F(q, \sigma, \tau, 0) = \langle d, 1\rangle$$

and similarly for $p \Vdash_{\mathbb{P}} \sigma = \tau$.

[4] More precisely, if $\pi : \mathbb{P} \to \mathbb{Q}$ is a dense embedding and G is $\mathbb{Q}$-generic over $\mathbb{M}$, then $M[G] = M[\pi^{-1}[G] \cap \mathbb{P}]$.
[5] See Definition 2.9.
[6] If for example $F(p, \sigma, \tau, 0) = \langle d, 1\rangle$, then for all $q \in d$, $q \Vdash_{\mathbb{P}} \sigma \in \tau$.

We are left with defining such a function F by induction on $\langle \text{rank}(\sigma)+\text{rank}(\tau), \text{rank}(\sigma)\rangle$, ordered lexicographically. If $\text{rank}(\sigma)+\text{rank}(\tau) = 0$, we simply put $F(p,\sigma,\tau,0) = \langle \{p\}, 0\rangle$ and $F(p,\sigma,\tau,1) = \langle \{p\}, 1\rangle$. Suppose now that $\text{rank}(\sigma) + \text{rank}(\tau) > 0$. We start with defining $F(p,\sigma,\tau,0)$. By induction, we may assume that for all $\pi \in \text{dom}(\tau)$ and for all $q \in \mathbb{P}$, $F(q,\sigma,\pi,1)$ has already been defined. There are two cases:

Case 1. There exist $\langle \pi, r\rangle \in \tau$ and $q \leq_{\mathbb{P}} p, r$ such that $F(q,\sigma,\pi,1) = \langle d, 1\rangle$ for some $d \in M$. Let $\alpha \in \text{Ord}^M$ be the minimal C-rank of such a set d. Then put $F(p,\sigma,\tau,0) = \langle e, 1\rangle$ where

$$e = \bigcup\{d \in C_{\alpha+1} \mid \exists\langle \pi, r\rangle \in \tau\, \exists q \leq_{\mathbb{P}} p, r\, F(q,\sigma,\pi,1) = \langle d, 1\rangle\}.$$

Case 2. Suppose we are not in Case 1. For each $\langle \pi, r\rangle \in \tau$, consider

$$D_{\pi,r} = \bigcup\{d \in M \mid \exists q \leq_{\mathbb{P}} p, r\, F(q,\sigma,\pi,1) = \langle d, 0\rangle\} \cup \{q \leq_{\mathbb{P}} p \mid q \perp_{\mathbb{P}} r\}.$$

We show that each $D_{\pi,r}$ is dense below p. Let $q \leq_{\mathbb{P}} p$. We want to find $s \leq_{\mathbb{P}} q$ in $D_{\pi,r}$. If $q \perp_{\mathbb{P}} r$ then we are done. Otherwise take $s \leq_{\mathbb{P}} q, r$. Since we are not in Case 1, $F(s,\sigma,\pi,1) = \langle d, 0\rangle$ for some $d \in M \setminus \{\emptyset\}$. Since d is nonempty, we may pick some condition $t \in d$. Then $t \in D_{\pi,r}$ and $t \leq_{\mathbb{P}} s \leq_{\mathbb{P}} q$.

By pretameness, there are conditions $q \leq_{\mathbb{P}} p$ and $\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \tau\rangle \in M$ such that each $d_{\pi,r}$ is a subset of $D_{\pi,r}$ which is predense below q. Let $\alpha \in \text{Ord}^M$ be minimal such that there is such q in $C_{\alpha+1}$. Then put $F(p,\sigma,\tau,0) = \langle e, 0\rangle$ where

$$e = \{q \in C_{\alpha+1} \cap \mathbb{P} \mid \exists\, \langle d_{\pi,r} \mid \langle \pi, r\rangle \in \tau\rangle \in M\, [\text{each } d_{\pi,r} \subseteq D_{\pi,r} \text{ is predense below } q]\}.$$

Now we define $F(p,\sigma,\tau,1)$. Again, we may inductively assume that for every $\pi \in \text{dom}(\sigma \cup \tau)$ and for every $q \in \mathbb{P}$, $F(q,\pi,\sigma,0)$ and $F(q,\pi,\tau,0)$ have already been defined. As above, we make a case distinction:

Case 1. There exist $\langle \pi, r\rangle \in \sigma \cup \tau$, a condition $q \in \mathbb{P}$ that is stronger than both p and r, $i \in 2$, $d, e \in M$ and $s \in d$ such that $F(q,\pi,\sigma,0) = \langle d, i\rangle$ and $F(s,\pi,\tau,0) = \langle e, 1-i\rangle$. Then let $\alpha \in \text{Ord}^M$ be the minimal C-rank of such a set e. Let $F(p,\sigma,\tau,1) = \langle f, 0\rangle$, where

$$f = \bigcup\{e \in C_{\alpha+1} \mid \exists\langle \pi, r\rangle \in \sigma \cup \tau\, \exists q \leq_{\mathbb{P}} p, r\, \exists i \in 2\, \exists d \in M\, \exists s \in d$$
$$[F(q,\pi,\sigma,0) = \langle d, i\rangle \wedge F(s,\pi,\tau,0) = \langle e, 1-i\rangle]\}.$$

Case 2. Suppose that we are not in Case 1. For each $\langle \pi, r\rangle \in \sigma \cup \tau$ let

$$D_{\pi,r} = \bigcup\{e \mid \exists q \leq_{\mathbb{P}} r\, \exists i \in 2\, \exists d\, \exists s \in d\, [F(q,\pi,\sigma,0) = \langle d, i\rangle \wedge F(s,\pi,\tau,0) = \langle e, i\rangle]\}$$
$$\cup \{q \in \mathbb{P} \mid q \perp_{\mathbb{P}} r\}.$$

Since Case 1 fails, each $D_{\pi,r}$ is dense below p. By pretameness there exist $q \leq_{\mathbb{P}} p$ and $\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \sigma \cup \tau\rangle \in M$ such that each $d_{\pi,r} \subseteq D_{\pi,r}$ is predense below q. Let $\alpha \in \text{Ord}^M$ be the least C-rank of such a condition q. Then put $F(p,\sigma,\tau,0) = \langle f, 1\rangle$ for

$$f = \{q \in C_{\alpha+1} \cap \mathbb{P} \mid \exists\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \sigma \cup \tau\rangle \in M\, [\text{each } d_{\pi,r} \subseteq D_{\pi,r} \text{ is predense below } q]\}.$$

This finishes the construction of F. It remains to check that F satisfies our desired properties. We proceed by induction. Suppose that $F(p,\sigma,\tau,0) = \langle e, 1\rangle$. We have to verify that for every $q \in e$, $q \Vdash_{\mathbb{P}} \sigma \in \tau$. Take $q \in e$ and a $\mathbb{P}$-generic filter G with $q \in G$. Since we are in Case 1, there is $\langle \pi, r\rangle \in \tau$ and $s \leq_{\mathbb{P}} p, r$ with $F(s,\sigma,\pi,1) = \langle d, 1\rangle$ for some d and $q \in d$. Then $q \leq_{\mathbb{P}} s$ and so $s \in G$. But by induction, since $\text{rank}(\pi) < \text{rank}(\tau)$, $q \Vdash_{\mathbb{P}} \sigma = \pi$ and so $\sigma^G = \pi^G \in \tau^G$.

Secondly, assume that $F(p,\sigma,\tau,0) = \langle e, 0\rangle$ and let $q \in e$ and G be $\mathbb{P}$-generic over $\mathbb{M}$ with $q \in G$. Now by Case 2 there is a sequence $\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \tau\rangle$ of sets $d_{\pi,r} \subseteq D_{\pi,r}$ which are predense below q. Suppose for a contradiction that $M[G] \models \sigma^G \in \tau^G$. Then there is $\langle \pi, r\rangle \in \tau$ with $r \in G$ and $\sigma^G = \pi^G$. Since $d_{\pi,r}$ is predense below q there is $s \in d_{\pi,r} \cap G$. Then s is compatible with r and so there are $d \in M$ and $t \leq_{\mathbb{P}} r$ with $F(t,\sigma,\pi,1) = \langle d, 0\rangle$ and $s \in d$. By induction, $s \Vdash_{\mathbb{P}} \sigma \neq \pi$, contradicting that $\sigma^G = \pi^G$. The proof that $F(p,\sigma,\tau,1)$ is as desired follows in a similar way. □

In [HKL$^+$, Theorem 1.3], it is shown that the forcing theorem may fail for notions of class forcing. The forcing used to prove this adds a binary predicate E on ω so that $\langle \omega, E\rangle$ isomorphic to $\langle M, \in\rangle$. The idea is that if the forcing theorem was satisfied one would obtain a first-order truth predicate in the ground model. In this section, we will prove a density version of this theorem. The following easy observation will be a key ingredient for our proof.

Lemma 2.2. *Suppose that $\mathbb{M}$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. Then $\mathbb{P}$ is pretame if and only if there exist no M-cardinal κ, $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}^M$ is cofinal".*

Proof. Suppose first that $\mathbb{P}$ is pretame. If $p \Vdash_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}^M$ is cofinal", consider

$$D_\alpha = \{q \leq_{\mathbb{P}} p \mid \exists \gamma \in \mathrm{Ord}^M \, q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}\}$$

for $\alpha < \kappa$. By pretameness there are $q \leq_{\mathbb{P}} p$ and sets $d_\alpha \subseteq D_\alpha$ in M which are predense below q. Now let

$$\beta = \sup\{\gamma + 1 \mid \gamma \in \mathrm{Ord}^M \wedge \exists \alpha < \kappa \, \exists r \in d_\alpha \; r \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}\}.$$

Then $q \Vdash_{\mathbb{P}} \mathrm{ran}(\dot{F}) \subseteq \check{\beta}$, a contradiction.

Conversely, suppose that $\langle D_i \mid i \in I\rangle \in \mathcal{C}$ with $I \in M$ is a sequence of dense subclasses of $\mathbb{P}$ and $p \in \mathbb{P}$ is such that there exist no $q \leq_{\mathbb{P}} p$ and $\langle d_i \mid i \in I\rangle \in M$ with each $d_i \subseteq D_i$ predense below q. Using Choice, we may assume that $I = \kappa$ is a cardinal in M. Let $\langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$ be a hierarchy on M. Now let G be $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$. In $\mathbb{M}[G]$, let $F : \kappa \to \mathrm{Ord}^M$ be the function defined by

$$F(\alpha) = \min\{\gamma \in \mathrm{Ord}^M \mid C_\gamma \cap D_\alpha \cap G \neq \emptyset\}.$$

Using the forcing theorem and [HKS, Observation 2.3], we may choose a name $\dot{F} \in \mathcal{C}$ for F and a condition $q \leq_{\mathbb{P}} p$ in G such that the above property of $\dot{F}$ is forced by q. But then q forces that $\dot{F}$ is cofinal in the ordinals – otherwise we could strengthen q to some r which forces the range of $\dot{F}$ to be contained in some ordinal γ and so $d_\alpha = D_\alpha \cap C_\gamma$ would be predense below r for every $\alpha \in I$, contradicting our assumption. □

The next step is to strengthen Lemma 2.2.

Lemma 2.3. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- and that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. Assume that $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ are such that $p \Vdash_{\mathbb{P}}$ "$\dot{F} \colon \check{\kappa} \to \mathrm{Ord}^M$ is cofinal" for some M-cardinal κ. Then there is a class name $\dot{E} \in \mathcal{C}$ and $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}}$ "$\dot{E} \colon \check{\kappa} \to \mathrm{Ord}^M$ is surjective."*

Proof. Since $\mathbb{P}$ satisfies the forcing theorem,

$$A = \{\langle r, \alpha, \beta\rangle \mid \exists s \leq_{\mathbb{P}} r \; s \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta}\} \in \mathcal{C}.$$

Hence there is a sequence $C = \langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle \in \mathcal{C}$ such that each C_α is of the form

$$A_{r,\alpha} = \{\beta \in \mathrm{Ord}^M \mid \exists s \leq_{\mathbb{P}} r \; s \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta}\}$$

for some $r \in \mathbb{P}$ and $\alpha \in \mathrm{Ord}^M$ such that $A_{r,\alpha}$ is a proper class, and moreover each such class $A_{r,\alpha}$ appears unboundedly often in C.

Claim 1. *There is a class $D = \langle D_\beta \mid \beta \in \mathrm{Ord}^M\rangle$ such that the classes D_β form a partition of Ord^M and $C_\alpha \cap D_\beta$ is a proper class for all $\alpha, \beta \in \mathrm{Ord}^M$.*

Proof. Let $k : \mathrm{Ord}^M \times \mathrm{Ord}^M \to \mathrm{Ord}^M$ be a bijection in $\mathcal{C}$ such that whenever $\bar{\gamma} < \gamma$, $k(\beta, \bar{\gamma}) < k(\beta, \gamma)$. Now we recursively define sets of ordinals $D^\gamma_\beta \in M$ in the following way: We start with $D^0_0 = \emptyset$. Let $\alpha, \beta, \gamma \in \mathrm{Ord}^M$ be such that $\alpha = k(\beta, \gamma)$ and assume that for all $\bar{\beta}, \bar{\gamma}$ with $k(\bar{\beta}, \bar{\gamma}) < \alpha$, $D^{\bar{\gamma}}_{\bar{\beta}}$ has already been defined. Now let $D^\gamma_\beta = \bigcup_{\bar{\gamma} < \gamma} D^{\bar{\gamma}}_\beta \cup \{\delta\}$, where δ is the least ordinal in $C_\gamma \setminus \bigcup\{D^{\bar{\gamma}}_{\bar{\beta}} \mid k(\bar{\beta}, \bar{\gamma}) < \alpha\}$. Finally, put $D_\beta = \bigcup_{\gamma \in \mathrm{Ord}^M} D^\gamma_\beta$ for each $\beta \in \mathrm{Ord}^M$. By construction, if $\beta \neq \bar{\beta}$ then D_β and $D_{\bar{\beta}}$ are disjoint. Moreover, since C_α appears unboundedly often in the enumeration defined above, $C_\alpha \cap D_\beta$ is a proper class for all $\alpha, \beta \in \mathrm{Ord}^M$. □

Suppose that D is a class as in the statement of the previous claim. If G is $\mathbb{P}$-generic over $\mathbb{M}$ with $p \in G$, let $E \colon \kappa \to \mathrm{Ord}^M$ be the function given by $E(\alpha) = \beta$ if $\dot{F}^G(\alpha) \in D_\beta$. Since $\mathbb{P}$ satisfies the forcing theorem, by [HKS, Observation 2.3], there is a class name $\dot{E} \in \mathcal{C}$ and a condition $q \in G$ below p which forces that $\dot{E}$ satisfies this definition.

Claim 2. *$q \Vdash_{\mathbb{P}}$ "$\dot{E}\colon \check{\kappa} \to \mathrm{Ord}^M$ is surjective".*

Proof. Suppose the contrary. Since $\mathbb{P}$ satisfies the forcing theorem, there is a condition $r \leq_{\mathbb{P}} q$ and some ordinal β such that $r \Vdash_{\mathbb{P}} \dot{E}(\check{\alpha}) \neq \check{\beta}$ for all $\alpha < \kappa$. Then there is $\alpha < \kappa$ such that $A_{r,\alpha} = \{\beta \in \mathrm{Ord}^M \mid \exists s \leq_{\mathbb{P}} r \ s \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\beta}\}$ is a proper class, since otherwise r forces that the range of $\dot{F}$ is bounded in Ord^M, contradicting our assumption on $\dot{F}$. By the previous claim, $A_{r,\alpha} \cap D_\beta$ is nonempty. Choose $\gamma \in A_{r,\alpha} \cap D_\beta$ and $s \leq_{\mathbb{P}} r$ so that $s \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$. Then $s \Vdash_{\mathbb{P}} \dot{E}(\check{\alpha}) = \check{\beta}$, contradicting our choice of r and of β. □

This completes the proof of Lemma 2.3. □

The next lemma states that we can modify the surjective function provided by Lemma 2.3 to a bijective function from some ordinal to M.

Lemma 2.4. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M, and that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. If there are $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ and $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}^M$ is surjective" for some M-cardinal κ, then there is an M-cardinal λ and $\dot{E} \in \mathcal{C}^{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{E} : \check{\lambda} \to \check{M}$ is bijective".*

Proof. Let κ be the least M-cardinal such that there is a condition $p \in \mathbb{P}$ with

$$p \Vdash_{\mathbb{P}} \text{“}\dot{F} : \check{\kappa} \to \mathrm{Ord}^M \text{ is surjective”}.$$

We define a class name $\dot{E} \in \mathcal{C}^{\mathbb{P}}$ such that p forces $\dot{E}$ to be a bijection between some ordinal $\gamma \leq \kappa$ and Ord^M. Namely, let $\dot{E}$ be

$$\left\{\langle \mathsf{op}(\check{\alpha}, \check{\beta}), q\rangle \mid q \Vdash_{\mathbb{P}} \text{“}\exists\delta\ \dot{D}(\delta) = \check{\beta} \text{ and } \delta \text{ is the } \check{\alpha}^{\text{th}} \text{ element of } \{\eta<\check{\kappa} \mid \forall\xi < \eta\ \dot{D}(\xi)\neq\dot{D}(\eta)\}\text{”}\right\},$$

where for $\sigma, \tau \in M^{\mathbb{Q}}$, $\mathsf{op}(\sigma,\tau)$ denotes the canonical name for the ordered pair $\langle \sigma^G, \tau^G\rangle$. Using minimality of κ, it follows that $\gamma = \kappa$. Since $\mathcal{C}$ contains a good well-order of M, we can further map Ord^M bijectively to M and obtain a name $\dot{F} \in \mathcal{C}^{\mathbb{P}}$ so that p forces $\dot{F}$ to be a bijection from κ to M. □

Definition 2.5. Let M be a model of ZF^-. A relation $T \subseteq \mathrm{Fml} \times M$ is a *first-order truth predicate for M* if

$$\langle \ulcorner\varphi\urcorner, x\rangle \in T \iff \langle M, \in\rangle \models \varphi(x)$$

holds for every $\ulcorner\varphi\urcorner \in \mathrm{Fml}$ and every $x \in M$, where Fml denotes the set of Gödel codes of $\mathcal{L}_\in$-formulae with one free variable.

Using the previous lemma, we are now ready to prove the main result of this section.

Theorem 2.6. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M, but no first-order truth predicate for M.[7] If $\mathbb{P}$ is a non-pretame notion of class forcing for $\mathbb{M}$, then there is a notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$ and a dense embedding $\pi\colon \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ such that $\mathbb{Q}$ does not satisfy the forcing theorem.*

Proof. Without loss of generality, we may assume that $\mathbb{P}$ satisfies the forcing theorem. Using Lemmata 2.2, 2.3 and 2.4, we can choose $p \in \mathbb{P}$ and a class name $\dot{F}$ such that

$$p \Vdash_{\mathbb{P}} \text{“}\dot{F} : \check{\kappa} \to \check{M} \text{ is bijective”}$$

for some M-cardinal κ.

We extend $\mathbb{P}$ to a forcing notion $\mathbb{Q}$ by adding suprema $p_{\alpha,\beta}$ for the classes

$$D_{\alpha,\beta} = \{q \leq_{\mathbb{P}} p \mid q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \dot{F}(\check{\beta})\}$$

for all $\alpha, \beta < \kappa$ such that $D_{\alpha,\beta}$ is nonempty. Let $X = \{\langle\alpha,\beta\rangle \in \kappa^2 \mid D_{\alpha,\beta} \neq \emptyset\}$. The following arguments generalize the proof of Theorem 1.3 in [HKL+]. Define

$$\dot{E} = \{\langle \mathsf{op}(\check{\alpha}, \check{\beta}), p_{\alpha,\beta}\rangle \mid \langle\alpha,\beta\rangle \in X\} \in M^{\mathbb{Q}}.$$

Assume for a contradiction that $\mathbb{Q}$ satisfies the forcing theorem. We will use $\dot{E}$ to show that $\mathcal{C}$ contains a first-order truth predicate for M, contradicting our assumptions.

[7] Note that by Tarski's theorem on the undefinability of truth, every model of the form $\langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ with a good well-order, where $\mathcal{C}$ only consists of the definable subsets of M, satisfies these requirements.

Claim 1. *Let G be $\mathbb{Q}$-generic over $\mathbb{M}$ with $p \in G$. In $M[G]$, $\langle M, \in \rangle$ is isomorphic to $\langle \kappa, \dot{E}^G \rangle$, witnessed by $\dot{F}^G$.*

Proof. By construction, $\dot{F}^G : \kappa \to M$ is a bijection. It remains to check that it defines an isomorphism. Let $\alpha, \beta < \kappa$ such that $\langle \alpha, \beta \rangle \in E$. Then $\langle \alpha, \beta \rangle \in X$ and $p_{\alpha,\beta} \in G$. But by definition of $\leq_{\mathbb{Q}}$, $p_{\alpha,\beta} \Vdash_{\mathbb{Q}} \dot{F}(\check{\alpha}) \in \dot{F}(\check{\beta})$ and therefore $\dot{F}^G(\alpha) \in \dot{F}^G(\beta)$ as desired. Conversely, suppose that $x \in y$ in M. Since $\dot{F}^G$ is surjective, there are $\alpha, \beta < \kappa$ such that $\dot{F}^G(\alpha) = x$ and $\dot{F}^G(\beta) = y$. Moreover, there must be $q \in G$ which forces that $\dot{F}(\check{\alpha}) \in \dot{F}(\check{\beta})$ and so $q \leq_{\mathbb{Q}} p_{\alpha,\beta}$. In particular, $p_{\alpha\beta} \in G$ and so $\langle \alpha, \beta \rangle \in E$. □

The next step will be to translate $\mathcal{L}_{\in}$-formulae into infinitary quantifier-free formulae in the forcing language of $\mathbb{Q}$, where $\in$ is translated to $\dot{E}$. The infinitary language $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{Q}, M)$ is built up from the atomic formulae $\check{q} \in \dot{G}$, $\sigma \in \tau$ and $\sigma = \tau$ for $q \in \mathbb{Q}$ and $\sigma, \tau \in M^{\mathbb{P}}$, the negation operator and set-sized conjunctions and disjunctions.[8]

Inductively, we assign to every $\mathcal{L}_{\in}$-formula φ with free variables in $\{v_0, \ldots, v_{k-1}\}$ and all $\vec{\alpha} = \alpha_0, \ldots, \alpha_{k-1} \in \kappa^k$ an $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{Q}, M)$-formula in the following way:

$$\begin{aligned}
(v_i = v_j)^*_{\vec{\alpha}} &= (\check{\alpha}_i = \check{\alpha}_j) \\
(v_i \in v_j)^*_{\vec{\alpha}} &= (\mathsf{op}(\check{\alpha}_i, \check{\alpha}_j) \in \dot{E}) \\
(\neg\varphi)^*_{\vec{\alpha}} &= (\neg\varphi^*_{\vec{\alpha}}) \\
(\varphi \vee \psi)^*_{\vec{\alpha}} &= (\varphi^*_{\vec{\alpha}} \vee \psi^*_{\vec{\alpha}}) \\
(\exists v_k \varphi)^*_{\vec{\alpha}} &= (\bigvee_{\beta < \kappa} \varphi^*_{\vec{\alpha},\beta}).
\end{aligned}$$

Note that by [HKL$^+$, Lemma 5.2], if $\mathbb{Q}$ satisfies the definability lemma for "$v_0 \in v_1$" or "$v_0 = v_1$", then it satisfies the forcing theorem for all infinitary formulae in the forcing language of $\mathbb{Q}$. The following claim will allow us to define a first-order truth predicate over M.

Claim 2. *For every $\mathcal{L}_{\in}$-formula φ with free variables among $\{v_0, \ldots, v_{k-1}\}$ and for all $\vec{x} = x_0, \ldots, x_{k-1}$ in M, the following statements are equivalent:*
(1) $M \models \varphi(\vec{x})$.
(2) $\forall \vec{\alpha} \in \kappa^k \, \forall q \leq_{\mathbb{P}} p \; q \Vdash_{\mathbb{P}}$ "$\forall i < k \; \dot{F}(\check{\alpha}_i) = \check{x}_i$" $\to q \Vdash_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$.
(3) $\exists \vec{\alpha} \in \kappa^k \, \exists q \leq_{\mathbb{P}} p \; q \Vdash_{\mathbb{P}}$ "$\forall i < k \; \dot{F}(\check{\alpha}_i) = \check{x}_i$" $\wedge \, q \Vdash_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$.

Proof. Observe that since $p \Vdash_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to M$ is bijective", (2) always implies (3).

We start with the atomic formula "$v_0 \in v_1$". Suppose first that $x \in y$ in M. Let $\alpha, \beta < \kappa$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \wedge \dot{F}(\check{\beta}) = \check{y}$. Take a $\mathbb{Q}$-generic filter with $q \in G$. Since $q \leq_{\mathbb{Q}} p_{\alpha,\beta}$, $p_{\alpha,\beta} \in G$. Moreover, $\langle \alpha, \beta \rangle \in \dot{E}^G$, so (2) holds. Assume now that (3) holds, i.e. there are $\alpha, \beta < \kappa$ and $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \wedge \dot{F}(\check{\beta}) = \check{y}$ and $q \Vdash_{\mathbb{Q}} (v_0 \in v_1)^*_{\alpha,\beta}$. Let G be $\mathbb{Q}$-generic with $q \in G$. Then by assmption $\langle \alpha, \beta \rangle \in \dot{E}^G$ and so $p_{\alpha,\beta} \in G$. In particular, this means that $x = \dot{F}^G(\alpha) \in \dot{F}^G(\beta) = y$. The proof for "$v_0 = v_1$" is similar.

Next we turn to negations. Suppose first that $M \models \neg\varphi(\vec{x})$ and let $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash_{\mathbb{P}} \forall i < k \, (\dot{F}(\check{\alpha}_i) = \check{x}_i)$. Assume, towards a contradiction, that $q \nVdash_{\mathbb{Q}} \neg\varphi^*_{\vec{\alpha}}$. Then there is $r \leq_{\mathbb{Q}} q$ with $r \Vdash_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$. By density, we may assume that $r \in \mathbb{P}$. Then $r \leq_{\mathbb{P}} p$ and so $\vec{\alpha}$ and r witness (3) for φ. By our inductive hypothesis we obtain that $M \models \varphi(\vec{x})$, a contradiction. The implication from (3) to (1) is similar.

Suppose now that $M \models (\varphi \vee \psi)(\vec{x})$. Without loss of generality, assume that $M \models \varphi(\vec{x})$. Now if $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash_{\mathbb{P}} \forall i < k \, (\dot{F}(\check{\alpha}_i) = \check{x}_i)$, by induction $q \Vdash_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$. But then in particular $q \Vdash_{\mathbb{Q}} (\varphi \vee \psi)^*_{\vec{\alpha}}$. In order to see that (3) implies (1), suppose that $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ witness (3). Then there must be a strengthenig $r \in \mathbb{Q}$ of q which satisfies, without loss of generality, $r \Vdash_{\mathbb{Q}} \varphi^*_{\vec{\alpha}}$. By density of $\mathbb{P}$ in $\mathbb{Q}$, we can assume that $r \in \mathbb{P}$. This means that $\vec{\alpha}$ and r witness that (3) holds for φ, so $M \models \varphi(\vec{x})$.

We are left with the existential case. Assume first that $M \models \exists v_k \varphi(\vec{x})$. Take $y \in M$ such that $M \models \varphi(\vec{x}, y)$ and let $\vec{\alpha} \in \kappa^k$ and $q \leq_{\mathbb{P}} p$ with $q \Vdash_{\mathbb{P}} \forall i < k \, (\dot{F}(\check{\alpha}_i) = \check{x}_i)$. Let now G be $\mathbb{Q}$-generic

[8] A detailed description of $\mathcal{L}^{\Vdash}_{\mathrm{Ord},0}(\mathbb{Q}, M)$ is given in [HKL$^+$, Section 5].

with $q \in G$. By an easy density argument there must be $r \leq_{\mathbb{P}} p$ and $\beta < \kappa$ with $r \in G$ and $r \Vdash_{\mathbb{P}} \dot{F}(\check{\beta}) = \check{y}$. By induction, $r \Vdash_{\mathbb{Q}} \varphi^*_{\vec{\alpha},\beta}$. In particular, $M[G] \models (\exists v_k \varphi)^*_{\vec{\alpha}}$. The converse follows in a similar way. □

Let Fml_1 denote the set of all Gödel codes of $\mathcal{L}_\in$-formulae whose only free variable is v_0. As a consequence of Claim 2, the class

$$T = \{\langle \ulcorner\varphi\urcorner, x\rangle \mid \ulcorner\varphi\urcorner \in \mathrm{Fml}_1 \wedge x \in M \wedge \forall \alpha < \kappa \forall q \leq_{\mathbb{P}} p\; q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{x} \rightarrow q \Vdash_{\mathbb{Q}} \varphi^*_\alpha\}$$

defines a first-order truth predicate for M, contradicting our assumptions on $\mathbb{M}$. □

Corollary 2.7. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M but no first-order truth predicate for M. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it densely satisfies the forcing theorem.* □

Furthermore, the proof of Theorem 2.6 yields the following.

Corollary 2.8. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M but no first-order truth predicate for M. Then for any M-cardinal κ, there is a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem, such that there is a κ-sequence S of subclasses of $\mathbb{P}$, for which $\mathbb{P}_S$ does not satisfy the forcing theorem.* □

The definition of the forcing relation in the existential case uses that $p \Vdash_{\mathbb{P}} \exists x \varphi(x)$ if and only if the class of all $q \leq_{\mathbb{P}} p$ such that there is a $\mathbb{P}$-name σ with $q \Vdash_{\mathbb{P}} \varphi(\sigma)$ is dense below p. The maximality principle states that (in set forcing) it is in fact not necessary to strengthen p in order to obtain a witness for an existential formula. We observe that for notions of class forcing which satisfy the forcing theorem, this principle is equivalent to the Ord-cc over models of GBC.

Definition 2.9. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem is said to satisfy the *maximality principle over* $\mathbb{M}$ if whenever $p \Vdash_{\mathbb{P}} \exists x \varphi(x, \vec{\sigma}, \vec{\Gamma})$ for some $p \in \mathbb{P}$, some $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_m, \vec{\Gamma})$ with class name parameters $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ and $\vec{\sigma}$ in $(M^{\mathbb{P}})^m$, then there is $\tau \in M^{\mathbb{P}}$ such that $p \Vdash_{\mathbb{P}} \varphi(\tau, \vec{\sigma}, \vec{\Gamma})$.

Lemma 2.10. *Assume that $\mathbb{M}$ is a model of GBC and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem. Then $\mathbb{P}$ satisfies the maximality principle if and only if it satisfies the* Ord*-cc over* $\mathbb{M}$.

Proof. Suppose first that $\mathbb{P}$ satisfies the maximality principle and let $A \in \mathcal{C}$ be an antichain in $\mathbb{P}$. Since $\mathcal{C}$ contains a well-ordering of M, we can extend A to a maximal antichain $A' \in \mathcal{C}$. It is enough to show that $A' \in M$. Clearly, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \exists x\, (x \in \check{A}' \cap \dot{G})$. Using the maximality principle, we obtain $\sigma \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \in \check{A}' \cap \dot{G}$. But then since $\mathsf{rnk}(\sigma^G) \leq \mathsf{rnk}(\sigma)$ for every $\mathbb{P}$-generic filter G, $A' \subseteq \mathbb{P} \cap (\mathsf{V}_\alpha)^M$ for $\alpha = \mathsf{rnk}(\sigma)$ and so $A' \in M$.

Conversely, assume that $\mathbb{P}$ satisfies the Ord-cc over $\mathbb{M}$ and let $p \Vdash_{\mathbb{P}} \exists x \varphi(x, \vec{\sigma}, \vec{\Gamma})$. Using the global well-order in $\mathcal{C}$ we can find an antichain $A \in \mathcal{C}$ which is maximal in $\{q \leq_{\mathbb{P}} p \mid \exists \sigma \in M^{\mathbb{P}}\; q \Vdash_{\mathbb{P}} \varphi(\sigma, \vec{\sigma}, \vec{\Gamma})\} \in \mathcal{C}$. Note that $\sup A = p$ and that $A \in M$ by assumption. For every $q \in A$, choose a name $\tau_q \in M^{\mathbb{P}}$ such that $q \Vdash_{\mathbb{P}} \varphi(\tau_q, \vec{\sigma}, \vec{\Gamma})$. Furthermore, for every $\mu \in \mathrm{dom}(\tau_q)$, let A^q_μ be a maximal antichain in $\{r \leq_{\mathbb{P}} q \mid \exists s\, \langle \mu, s\rangle \in \tau_q \wedge r \leq_{\mathbb{P}} s\}$. Now put

$$\sigma = \{\langle \mu, r\rangle \mid \exists q \in A\; \mu \in \mathrm{dom}(\tau_q) \wedge r \in A^q_\mu\}.$$

By construction, $q \Vdash_{\mathbb{P}} \sigma = \tau_q$ for every $q \in A$ and so $p \Vdash_{\mathbb{P}} \varphi(\sigma)$. □

3. Preservation of axioms

The theorem below is a version of a theorem of Sy Friedman [Fri00, Proposition 2.17 and Lemma 2.19], that we adjusted to our generalized setting.

Theorem 3.1. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of GB^- with a hierarchy witnessed by $\langle C_\alpha \mid \alpha \in \mathrm{Ord}^M\rangle$. Then the following statements hold for every notion of class forcing $\mathbb{P}$ for $\mathbb{M}$.*

1. *If $\mathbb{P}$ is pretame and G is $\mathbb{P}$-generic over $\mathbb{M}$ then $\mathbb{M}[G]$ satisfies GB^- and has a hierarchy. Moreover, if $\mathbb{M}$ satisfies global choice, then so does $\mathbb{M}[G]$.*
2. *Suppose that $\mathbb{M}$ has a hierarchy and that for every $p \in \mathbb{P}$ there is a $\mathbb{P}$-generic filter G such that $p \in G$ and Replacement holds in $\mathbb{M}[G]$. Then $\mathbb{P}$ is pretame.*

Proof. For (1), suppose that $\mathbb{P}$ is pretame and that G is $\mathbb{P}$-generic over $\mathbb{M}$. It is easy to check that $\mathbb{M}[G]$ satisfies all set axioms of GB^- except possibly for Separation, Collection and Union. Moreover, Collection implies Separation, and the preservation of Separation can easily be seen to imply the preservation of Union.

To see that $\mathbb{M}[G]$ satisfies Collection, assume that $\mathbb{M}[G] \models \forall x \in \sigma^G \, \exists y \; \varphi(x, y, \Gamma^G)$, where $\sigma \in M^{\mathbb{P}}$, $\Gamma \in \mathcal{C}^{\mathbb{P}}$ and φ is an $\mathcal{L}_\in$-formula with one class parameter. By the truth lemma there is $p \in G$ such that $p \Vdash_{\mathbb{P}} \forall x \in \sigma \, \exists y \, \varphi(x, y, \Gamma)$. For each $\langle \pi, r \rangle \in \sigma$, the class

$$D_{\pi,r} = \{s \in \mathbb{P} \mid [s \leq_{\mathbb{P}} p, r \wedge \exists \mu \in M^{\mathbb{P}} \, (s \Vdash_{\mathbb{P}} \varphi(\pi, \mu, \Gamma))] \vee s \perp_{\mathbb{P}} r\}$$

is dense below p in $\mathbb{P}$. By pretameness there is $q \in G$ which strengthens p and there are sets $d_{\pi,r} \subseteq D_{\pi,r}$ for each $\langle \pi, r \rangle \in \sigma$ such that each $d_{\pi,r} \in M$ is predense below q. Using Collection in $\mathbb{M}$, there is a set $x \in M$ such that for each $\langle \pi, r \rangle \in \sigma$ and for each $s \in d_{\pi,r}$ there is $\mu \in x$ such that $s \Vdash_{\mathbb{P}} \varphi(\pi, \mu, \Gamma)$. Now put

$$\tau = \{\langle \mu, s \rangle \mid \mu \in x \wedge \exists \langle \pi, r \rangle \in \sigma \, (s \in d_{\pi,r} \wedge s \Vdash_{\mathbb{P}} \varphi(\pi, \mu, \Gamma))\}.$$

By construction, $\mathbb{M}[G] \models \forall x \in \sigma^G \, \exists y \in \tau^G \, \varphi(x, y, \Gamma^G)$.

To see that $\mathbb{M}[G]$ satisfies first-order class comprehension, note that

$$\Gamma = \{\langle \sigma, p \rangle \mid p \Vdash_{\mathbb{P}}^{\mathbb{M}} \varphi(\sigma, \Gamma_0, \ldots, \Gamma_{n-1})\} \in \mathcal{C}^{\mathbb{P}}$$

is a class name for the class $\{x \mid \varphi(x, \Gamma_0^G, \ldots, \Gamma_{n-1}^G)\}$. Furthermore, we can define a hierarchy $\langle D_\alpha \mid \alpha \in \mathrm{Ord}^M \rangle$ in $\mathcal{C}[G]$ by

$$D_\alpha = \{x \in \mathbb{M}[G] \mid \exists \sigma \in M^{\mathbb{P}} \cap C_\alpha \; \sigma^G = x\} = \{\langle \sigma, \mathbb{1} \rangle \mid \sigma \in C_\alpha\}^G \in M[G]$$

for every $\alpha \in \mathrm{Ord}^M$. Finally, if $\prec$ is a global well-order of M in $\mathcal{C}$ then

$$x \lhd y \iff \exists \sigma \in M^{\mathbb{P}}[x = \sigma^G \wedge \forall \tau \in M^{\mathbb{P}}(y = \tau^G \to \sigma \prec \tau)]$$

defines a global well-order of $M[G]$ in $\mathcal{C}[G]$.

Now we turn to (2). Assume, towards a contradiction, that $\langle D_i \mid i \in I \rangle$ is a sequence of dense classes and p is a condition in $\mathbb{P}$ which witness that pretameness fails. By assumption there is a $\mathbb{P}$-generic filter G containing p such that $\mathbb{M}[G]$ satisfies Replacement. Now consider the function

$$F : I \to \mathrm{Ord}^M, F(i) = \min\{\alpha \in \mathrm{Ord}^M \mid G \cap D_i \cap C_\alpha \neq \emptyset\}.$$

Since $\mathbb{M}[G]$ satisfies Replacement, $\mathrm{ran}(F) \in M[G]$. Let $\gamma \in \mathrm{Ord}^M$ be such that $\mathrm{ran}(F) \subseteq \gamma$ and

$$D = \{q \leq_{\mathbb{P}} p \mid \exists i \in I \, \forall r \in D_i \cap C_\gamma \, (q \perp_{\mathbb{P}} r)\}.$$

By assumption, D is dense below p. Pick $q \in G \cap D$ and let $i \in I$ such that q is incompatible with all elements of $D_i \cap C_\gamma$. But then $F(i) > \gamma$, a contradiction. □

Corollary 3.2. *Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M, and let $\mathbb{P}$ be a notion of class forcing for $\mathbb{M}$. Then the following statements are equivalent:*

(1) $\mathbb{P}$ *is pretame.*
(2) $\mathbb{P}$ *preserves* GB^-*.*
(3) $\mathbb{P}$ *preserves Collection.*
(4) $\mathbb{P}$ *preserves Replacement.*
(5) $\mathbb{P}$ *preserves Separation and satisfies the forcing theorem.*

Proof. The implications from (2) to (3), from (3) to (4) and from (4) to (5) are trivial. That (1) implies (2) and that (4) implies (1) follows from Theorem 3.1. Finally, the implication from (5) to (4) is shown in [HKS, Theorem 5.1]. Note that this is the only time that we are using the fact that $\mathcal{C}$ contains a good well-order; for the other implications it suffices to require that $\mathbb{M}$ has a hierarchy. □

4. Boolean completions

In this section, $\mathbb{M} = \langle M, \mathcal{C}\rangle$ will always denote a countable transitive model of GB^-. As has been shown in [HKL+], the existence of Boolean completions is closely related to the forcing theorem. Namely by [HKL+, Theorem 5.5], if $\mathbb{M}$ has a hierarchy and $\mathbb{P}$ is a separative notion of class forcing for $\mathbb{M}$, then $\mathbb{P}$ has a Boolean M-completion iff it satisfies the forcing theorem for all $\mathcal{L}_\in$-formulae. Thus the following is a consequence of Theorem 2.7.

Theorem 4.1. *Suppose that $\mathcal{C}$ contains a good well-order of M, but no first-order truth predicate for M. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame over $\mathbb{M}$ if and only if it densely has a Boolean M-completion.* □

Lemma 4.2. *If a notion of class forcing $\mathbb{P}$ has a Boolean $\mathcal{C}$-completion $\mathbb{B}$, then it is unique. Moreover, if $\mathbb{P}$ has a unique Boolean M-completion $\mathbb{B}$ then $\mathbb{B}$ is a Boolean $\mathcal{C}$-completion of $\mathbb{P}$.*

Proof. The proof of the first statement is exactly as for set forcing. Suppose now that $\mathbb{B}$ is the unique Boolean M-completion of $\mathbb{P}$ and suppose for a contradiction that $A \subseteq \mathbb{B}$ is a class in $\mathcal{C}$ which does not have a supremum in $\mathbb{B}$. Let $\mathbb{Q}$ be the forcing notion obtained from $\mathbb{P}$ by adding $\sup A$. Then since $\mathbb{P}$ satisfies the forcing theorem, by Lemma 1.9 so does $\mathbb{Q}$ and hence $\mathbb{Q}$ has a Boolean M-completion $\mathbb{B}'$. But by our assumption, $\mathbb{B}$ and $\mathbb{B}'$ are isomorphic and hence $\sup A$ exists in $\mathbb{B}$, a contradiction. □

Definition 4.3. Suppose that $\mathbb{P}$ is a notion of class forcing for $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$. If $A, B \subseteq \mathbb{P}$ with $A, B \in \mathcal{C}$, we say that $\sup_\mathbb{P} A = \sup_\mathbb{P} B$ if

(1) A is predense below every $b \in B$ and
(2) B is predense below every $a \in A$.

Note that this definition is possible even if the suprema do not exist in $\mathbb{P}$. On the other hand, if $\sup A = \sup B$ and A has a supremum in $\mathbb{P}$ then so does B and indeed they coincide.

The following observation is a slight strengthening of a lemma which is essentially due to Joel Hamkins and appears within the proof of [HKL+, Theorem 9.4]. The below proof is very similar to the one appearing in [HKL+, Theorem 9.4], however we also improved our original presentation.

Lemma 4.4. *Suppose that $\mathcal{C}$ contains a good well-order of M. If $\mathbb{P}$ does not satisfy the* Ord*-cc, then there is an antichain $A \in \mathcal{C}$ such that for every $B \in M$ with $B \subseteq \mathbb{P}$, $\sup B \neq \sup A$. In particular, A does not have a supremum in $\mathbb{P}$.*

Proof. Let $A \in \mathcal{C}$ be a class-sized antichain in $\mathbb{P}$. We claim that there is a subclass of A in $\mathcal{C}$ which fulfills the desired properties. Suppose for a contradiction that no such subclass exists. Using the good well-order of M, we can assume that the domain of $\mathbb{P}$ is Ord^M. Let $\pi : \mathrm{Ord}^M \to A$ be a bijection in $\mathcal{C}$. Furthermore, there is an injection $\varphi : \mathcal{P}(\mathrm{Ord}^M) \cap M \to \mathrm{Ord}^M$ in $\mathcal{C}$. This gives us a mapping $i : \mathcal{P}(\mathrm{Ord}^M) \cap \mathcal{C} \to \mathrm{Ord}^M$ in V which maps $X \subseteq \mathrm{Ord}^M$ to $\varphi(B)$, where B is the least (with respect to our given global well-order) set $B \subseteq \mathbb{P}$ in M such that $\sup_\mathbb{P} \pi''X = \sup_\mathbb{P} B$. Since A is an antichain, i is injective. Moreover, whether $i(X) = \alpha$ is definable over $\mathbb{M}$, so

$$C = \{\alpha \in \mathrm{Ord}^M \mid \pi(\alpha) \not\leq_\mathbb{P} \alpha \wedge i(X_\alpha) = \alpha\}$$

is in $\mathcal{C}$ for $X_\alpha = \{\beta \in \mathrm{Ord}^M \mid \pi(\beta) \leq_\mathbb{P} \alpha\}$.

Claim 1. *For each $\alpha \in \mathrm{Ord}^M$ we have $\alpha \in C$ if and only if there is $X \in \mathcal{P}(\mathrm{Ord}^M) \cap \mathcal{C}$ such that $i(X) = \alpha$ and $\alpha \notin X$.*

Proof. Suppose first that $\alpha \in C$. Then $\alpha \notin X_\alpha$ and so we can choose $X = X_\alpha$. Conversely, suppose that $X \in \mathcal{P}(\mathrm{Ord}^M) \cap \mathcal{C}$ is such that $i(X) = \alpha$ and $\alpha \notin X$. Then $X = X_\alpha$, because $\pi''X$ and $\pi''X_\alpha$ are both subsets of the antichain A and have the same supremum. Hence $\alpha \in C$. □

We will use Claim 1 to derive a contradiction similar to Russell's paradox. Consider $\beta = i(C)$. If $\beta \in C$ then by Claim 1 there is X such that $i(X) = \beta$ but $\beta \notin X$. By injectivity of i, this means that $X = C$, a contradiction. On the other hand, it is also impossible that $\beta \notin C$, since otherwise $X = C$ would witness that $\beta \in C$. □

The following theorem characterizes the Ord-cc in terms of the existence of Boolean completions. Note that the equivalence of (1) and (2) is exactly the statement of [HKL$^+$, Theorem 9.4]. For the benefit of the reader, we nevertheless give a full proof.

Theorem 4.5. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M. Then the following statements are equivalent for every separative partial order $\mathbb{P}$:*

(1) $\mathbb{P}$ *satisfies the* Ord-*cc.*
(2) $\mathbb{P}$ *has a unique Boolean M-completion.*
(3) $\mathbb{P}$ *has a Boolean $\mathcal{C}$-completion.*

Proof. Suppose first that $\mathbb{P}$ satisfies the Ord-cc. Then $\mathbb{P}$ is pretame and hence it has a Boolean M-completion $\mathbb{B}$ by Theorem 4.1. Assume that $\mathbb{B}'$ is another Boolean M-completion. Without loss of generality, we may assume that the domain of $\mathbb{P}$ is a subset of the domains of $\mathbb{B}$ and $\mathbb{B}'$. Then we can define an isomorphism between $\mathbb{B}$ and $\mathbb{B}'$ by mapping $b \in \mathbb{B}$ to $\sup_{\mathbb{B}'} A$, where $A \in M$ is a maximal antichain in $\{p \in \mathbb{P} \mid p \leq_{\mathbb{B}} b\}$. The equivalence of (2) and (3) is a direct consequence of Lemma 4.2. To see that (3) implies (1), suppose that $\mathbb{B}$ is a Boolean $\mathcal{C}$-completion of $\mathbb{P}$. Assume, towards a contradiction, that $\mathbb{P}$ does not satisfy the Ord-cc. Then neither does $\mathbb{B}$. But then by Lemma 4.4, $\mathbb{B}$ cannot be $\mathcal{C}$-complete. □

5. The extension maximality principle

This section is motivated by the following easy observation which is mentioned in [HKL$^+$, Corollary 2.3]. The collapse forcing $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$, which consists of functions $n \to \mathrm{Ord}^M$ for $n \in \omega$, is dense in $\mathrm{Col}(\omega, \mathrm{Ord})^M$. However, unlike $\mathrm{Col}(\omega, \mathrm{Ord})^M$ which collapses all M-cardinals, the subforcing $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ does not add any new sets, so $\mathrm{Col}(\omega, \mathrm{Ord})^M$ and $\mathrm{Col}_*(\omega, \mathrm{Ord})^M$ do not have the same generic extensions. We will show that, under sufficient conditions on the ground model $\mathbb{M}$, the property of $\mathbb{P}$ of having the same generic extensions as all forcing notions into which $\mathbb{P}$ densely embeds is in fact equivalent to the pretameness of $\mathbb{P}$. Throughout this section, let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be countable transitive model of GB^-.

Definition 5.1. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ satisfies the

(1) *extension maximality principle (EMP)* over $\mathbb{M}$ if whenever $\mathbb{Q}$ is a notion of class forcing for $\mathbb{M}$ and $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding in $\mathcal{C}$ then for every $\mathbb{Q}$-generic filter G over $\mathbb{M}$, $M[G] = M[\pi^{-1}(G) \cap \mathbb{P}]$.
(2) *strong extension maximality principle (SEMP)* over $\mathbb{M}$ if whenever $\mathbb{Q}$ is a notion of class forcing for $\mathbb{M}$, $\pi\colon \mathbb{P} \to \mathbb{Q}$ is a dense embedding in $\mathcal{C}$ and $\sigma \in M^{\mathbb{Q}}$, then there is $\tau \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \pi(\tau)$.

Theorem 5.2. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of GB^- such that $\mathcal{C}$ contains a good well-order of M. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it satisfies the forcing theorem and the EMP.*

Proof. Suppose first that $\mathbb{P}$ is pretame. By Theorem 2.1, $\mathbb{P}$ satisfies the forcing theorem. Let $\mathbb{Q}$ be a notion of class forcing such that $\mathbb{P}$ embeds densely into $\mathbb{Q}$ and let G be $\mathbb{Q}$-generic over $\mathbb{M}$. Without loss of generality, we may assume that $\mathbb{P}$ is a dense suborder of $\mathbb{Q}$. Fix a $\mathbb{Q}$-name σ. We claim that $\sigma^G \in M[G \cap \mathbb{P}]$. For every $q \in \mathrm{tc}(\sigma) \cap \mathbb{Q}$, let $D_q = \{p \in \mathbb{P} \mid p \leq_{\mathbb{Q}} q \vee p \perp_{\mathbb{Q}} q\}$. Then D_q is a dense subclass of $\mathbb{P}$. By pretameness, there is $p \in G \cap \mathbb{P}$ and there are $d_q \subseteq D_q$ which are predense below p in $\mathbb{P}$. Now we define inductively, for every name τ in $\mathrm{tc}(\{\sigma\}) \cap M^{\mathbb{Q}}$,

$$\bar{\tau} = \{\langle \bar{\mu}, r\rangle \mid \exists s[\langle \mu, s\rangle \in \tau \wedge r \in d_s \wedge r \leq_{\mathbb{Q}} s]\}.$$

But then $\bar{\sigma} \in M^{\mathbb{P}}$ and $\sigma^G = \bar{\sigma}^{G \cap \mathbb{P}} \in M[G \cap \mathbb{P}]$.

Conversely, assume that $\mathbb{P}$ is not pretame but satisfies the forcing theorem. Then there is a $\mathbb{P}$-generic filter G such that Replacement fails in the generic extension $\mathbb{M}[G]$, and by [HKS, Theorem 5.1] so does Separation (note that this is where we use the assumption about the good well-order). By [HKS, Lemma 3.3] there are $\Gamma \in \mathcal{C}^{\mathbb{P}}$, $\sigma \in M^{\mathbb{P}}$ and $p \in G$ such that $p \Vdash_{\mathbb{P}} \Gamma \subseteq \sigma$ and there are no $q \in G$ and $\tau \in M^{\mathbb{P}}$ such that $q \Vdash_{\mathbb{P}} \Gamma = \tau$. For $\mu \in \mathrm{dom}(\sigma)$ consider

$$A_\mu = \{q \in \mathbb{P} \mid q \Vdash_{\mathbb{P}} \mu \in \Gamma\}.$$

Let $\mathbb{Q}$ be the forcing obtained from $\mathbb{P}$ by adding $\sup A_\mu$ for each $\mu \in \mathrm{dom}(\sigma)$ such that A_μ is nonempty below p, as described in Section 1. Without loss of generality, we may assume that $\mathbb{P}$ is a subset of $\mathbb{Q}$ and then $\mathbb{P}$ is actually a dense subset of $\mathbb{Q}$. Consider the $\mathbb{Q}$-name

$$\tau = \{\langle \mu, \sup A_\mu \rangle \mid \mu \in \mathrm{dom}(\sigma), A_\mu \text{ is nonempty below } p\} \in M^{\mathbb{Q}}.$$

Let H be the $\mathbb{Q}$-generic induced by G, that is the upwards closure of G in $\mathbb{Q}$. Then τ is a $\mathbb{Q}$-name for Γ^G, so $\tau^H = \Gamma^G \in M[H] \setminus M[G]$, proving the failure of the EMP. □

Note that in the proof of Theorem 5.2 we have only used the good well-order of M to show that every forcing notion which satisfies the forcing theorem and the EMP is pretame; for the other direction it suffices to assume that $\mathbb{M}$ has a hierarchy.

Example 5.3. *Jensen coding* $\mathbb{P}$ (see [BJW82]) is a pretame notion of class forcing which over a model M of ZFC adds a generic real x such that the $\mathbb{P}$-generic extension is of the form $\mathsf{L}[x]$. Moreover, there is a class name Γ for x such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} M[\dot{G}] = \mathsf{L}[\Gamma]$, but there is no set name σ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma = \Gamma$. Let $\mathbb{Q}$ be the forcing notion obtained from Jensen coding by adding the suprema $p_n = \sup\{p \in \mathbb{P} \mid p \Vdash_{\mathbb{P}} \check{n} \in \Gamma\}$. Since $\mathbb{P}$ is pretame and dense in $\mathbb{Q}$, it follows that $\mathbb{Q}$ is also pretame. By Theorem 5.2, $\mathbb{P}$ satisfies the EMP and hence $\mathbb{P}$ and $\mathbb{Q}$ produce the same generic extensions. In particular, this means that if G is $\mathbb{Q}$-generic then $M[G \cap \mathbb{P}] = M[G] = \mathsf{L}[\sigma^G]$, where $\sigma = \{\langle \check{n}, p_n \rangle \mid n \in \omega\} \in M^{\mathbb{Q}}$.

Lemma 5.4. *Suppose that $\mathbb{M} = \langle M, \mathcal{C} \rangle$ is a model of GB^- and that $\mathcal{C}$ contains a good well-order of M. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ satisfies the SEMP if and only if it satisfies the* Ord*-cc over $\mathbb{M}$.*

Proof. Suppose first that $\mathbb{P}$ satisfies the Ord-cc. Suppose that there is a dense embedding from $\mathbb{P}$ into some forcing notion $\mathbb{Q}$. Without loss of generality, we may assume that $\mathbb{P}$ is a suborder of $\mathbb{Q}$. We prove by induction on $\mathrm{rank}(\sigma)$ that for every $\sigma \in M^{\mathbb{Q}}$ there is $\bar{\sigma} \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \bar{\sigma}$. Assume that this holds for all τ of rank less than $\mathrm{rank}(\sigma)$. Then for every $\tau \in \mathrm{dom}(\sigma)$ there is $\bar{\tau} \in M^{\mathbb{P}}$ with $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \tau = \bar{\tau}$. For each condition $q \in \mathrm{range}(\sigma)$, let $D_q = \{p \in \mathbb{P} \mid p \leq_{\mathbb{Q}} q\}$ and choose an antichain A_q which is maximal in D_q. By assumption, we may do this so that $\langle\langle q, A_q \rangle \mid q \in \mathrm{range}(\sigma)\rangle \in M$. Then put

$$\bar{\sigma} = \{\langle \bar{\tau}, p \rangle \mid \exists q \in \mathbb{Q}\, [\langle \tau, q \rangle \in \sigma \wedge p \in A_q]\} \in M^{\mathbb{P}}.$$

By construction, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \bar{\sigma}$.

Conversely, suppose that $\mathbb{P}$ does not satisfy the Ord-cc. Then by Lemma 4.4 there is an antichain $A \in \mathcal{C}$ such that for no $B \in M$ with $B \subseteq \mathbb{P}$, $\sup_{\mathbb{P}} A = \sup_{\mathbb{P}} B$. Let $\mathbb{Q} = \mathbb{P} \cup \{\sup A\}$ be the extension of $\mathbb{P}$ given by adding the supremum of A. Now consider $\sigma = \{\langle \check{0}, \sup A \rangle\} \in M^{\mathbb{Q}}$. We claim that σ witnesses the failure of the SEMP. Suppose for a contradiction that there is $\tau \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \tau$. Let $\tau = \{\langle \mu_i, p_i \rangle \mid i \in I\}$ for some $I \in M$. But then it is easy to check that $\sup_{\mathbb{P}}\{p_i \mid i \in I\} = \sup_{\mathbb{P}} A$, contradicting our assumption on A. □

6. Nice Names

This section is motivated by the observation in Lemma 6.2 below, namely that - unlike in the context of set forcing - there are sets of ordinals in class-generic extensions which do not have a nice name. We will characterize both pretameness and the Ord-cc in terms of the existence of nice names for sets of ordinals. We assume that $\mathbb{M} = \langle M, \mathcal{C} \rangle \models \mathsf{GB}^-$ throughout this section, however for the characterization of pretameness, we will in fact need to work over a model of KM.

Definition 6.1. Let $\mathbb{P}$ be a notion of class forcing. A name $\sigma \in M^{\mathbb{P}}$ for a set of ordinals is a *nice name* if it is of the form $\bigcup_{\alpha<\gamma}\{\check{\alpha}\} \times A_\alpha$ for some $\gamma \in \mathrm{Ord}^M$, where each $A_\alpha \in M$ is a set-sized antichain of conditions in $\mathbb{P}$.

Lemma 6.2. *Let $\mathbb{M}$ be a model of GB^- and let $\mathbb{P}$ denote $\mathrm{Col}(\omega, \mathrm{Ord})^M$. Then in every $\mathbb{P}$-generic extension there is a subset of ω which does not have a nice $\mathbb{P}$-name.*

Proof. Consider the canonical name $\sigma = \{\langle \check{n}, \{\langle n, 0 \rangle\}\rangle \mid n \in \omega\}$ for the set of natural numbers which are mapped to 0 by the generic function from ω to the ordinals. Let G be $\mathbb{P}$-generic over

$\mathbb{M}$. We show that the complement of σ^G is an element of $M[G]$, but does not have a nice $\mathbb{P}$-name in M. Suppose for a contradiction that there are $p \in G$ and a nice $\mathbb{P}$-name

$$\tau = \bigcup_{n\in\omega} \{\check{n}\} \times A_n \in M,$$

where each $A_n \in M$ is an antichain, such that $p \Vdash_{\mathbb{P}} \check{\omega} \setminus \sigma = \tau$. Let $n \in \omega \setminus \operatorname{dom}(p)$ and choose $\alpha > \sup\{r(i) \mid r \in A_n \wedge i \in \operatorname{dom}(r)\}$. Then $q = p \cup \{\langle n, \alpha\rangle\}$ strengthens p and $q \Vdash_{\mathbb{P}} \check{n} \in \tau$. Hence there must be some $r \in A_n$ which is compatible with q. But then $n \notin \operatorname{dom}(r)$, so p and $r \cup \{\langle n, 0\rangle\}$ are compatible. Let $s \leq_{\mathbb{P}} p, r \cup \{\langle n, 0\rangle\}$ witness this. Then $s \Vdash_{\mathbb{P}} \check{n} \in \sigma \cap \tau$, a contradiction.

That the complement of σ^G has a $\mathbb{P}$-name in M and is thus an element of $M[G]$ follows from [HKS, Lemma 8.7]. For the benefit of the reader, we will provide an even shorter proof for the present special case. For each $n \in \omega$, consider the $\mathbb{P}$-name

$$\tau_n = \check{n} \cup \{\langle \check{m}, \{\langle i, 0\rangle \mid n \leq i < m\}\rangle \mid m > n\}.$$

Then each τ_n is a name for the least $k \geq n$ such that $k \notin \sigma^G$. Now put $\tau = \{\langle \tau_n, \mathbb{1}_{\mathbb{P}}\rangle \mid n \in \omega\}$. Since by an easy density argument the complement of σ^G is unbounded in ω, τ is as desired. □

Definition 6.3. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is said to be

(1) *nice* if for every $\gamma \in \mathrm{Ord}^M$, for every $\sigma \in M^{\mathbb{P}}$ and for every $\mathbb{P}$-generic filter G such that $\sigma^G \subseteq \gamma$ there is a nice name $\tau \in M^{\mathbb{P}}$ such that $\sigma^G = \tau^G$.
(2) *very nice* if for every $\gamma \in \mathrm{Ord}^M$ and for every $\sigma \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \subseteq \check{\gamma}$ there is a nice name $\tau \in M^{\mathbb{P}}$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma = \tau$.

Example 6.4. (1) By Lemma 6.2, $\mathrm{Col}(\omega, \mathrm{Ord})^M$ is not nice.

(2) Suppose that $\mathcal{C}$ contains a good well-order of M. Then every pretame notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is nice. To see this, let $\gamma \in \mathrm{Ord}^M$ be an ordinal and let $p \in \mathbb{P}$ and $\sigma \in M^{\mathbb{P}}$ be such that $p \Vdash_{\mathbb{P}} \sigma \subseteq \check{\gamma}$.[9] For each $\alpha < \gamma$, consider the class

$$D_\alpha = \{q \leq_{\mathbb{P}} p \mid q \Vdash_{\mathbb{P}} \alpha \in \sigma \vee q \Vdash_{\mathbb{P}} \alpha \notin \sigma\} \in \mathcal{C},$$

which is dense below p. By pretameness there exist $q \leq_{\mathbb{P}} p$ and for every $\alpha < \gamma$ a set $d_\alpha \subseteq D_\alpha$ in M which is predense below q. For every $\alpha < \gamma$, choose an antichain $a_\alpha \subseteq d_\alpha$ which is maximal in d_α, and let $A_\alpha = \{r \in a_\alpha \mid r \Vdash_{\mathbb{P}} \check{\alpha} \in \sigma\}$. Then

$$\tau = \bigcup_{\alpha<\gamma} \{\check{\alpha}\} \times A_\alpha \in M^{\mathbb{P}}$$

is a nice name for a subset of γ and $q \Vdash_{\mathbb{P}} \sigma = \tau$.

(3) If $\mathcal{C}$ contains a good well-order of M, then every notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the Ord-cc is very nice: Suppose that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma \subseteq \check{\gamma}$. For every $\alpha < \gamma$, we can choose an antichain A_α which is maximal in $\{q \in \mathbb{P} \mid \exists\langle \mu, p\rangle \in \sigma\, [q \leq_{\mathbb{P}} p \wedge q \Vdash_{\mathbb{P}} \mu = \check{\alpha}]\}$. Since $\mathbb{P}$ satisfies the Ord-cc, making use of the global well-order we can do this so that $\langle\langle \alpha, A_\alpha\rangle \mid \alpha < \gamma\rangle \in M$. Then

$$\tau = \bigcup_{\alpha<\gamma} \{\check{\alpha}\} \times A_\alpha \in M^{\mathbb{P}}$$

is a nice name and it is easy to check that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \sigma = \tau$.

(4) Every M-complete Boolean algebra $\mathbb{B}$ is very nice, since we can always define *Boolean values* $[\![\varphi]\!]_{\mathbb{B}}$ for quantifier-free infinitary formulae φ which mention only set names (see [HKL$^+$, Theorem 5.5]). More precisely, if $\sigma \in M^{\mathbb{B}}$ such that $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}} \sigma \subseteq \check{\gamma}$ for some ordinal γ, the name

$$\tau = \{\langle \check{\alpha}, [\![\check{\alpha} \in \sigma]\!]_{\mathbb{B}}\rangle \mid \alpha < \gamma\} \in M^{\mathbb{B}}$$

is a nice name so that $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}} \sigma = \tau$. In particular, this shows that there are very nice notions of class forcing which are not pretame (for example the Boolean M-completion of $\mathrm{Col}(\omega, \mathrm{Ord})^M$), since every notion of class forcing for $\mathbb{M}$ which satisfies the forcing theorem has a Boolean M-completion by [HKL$^+$, Theorem 5.5].

[9]Note that using the good wellorder of M, it follows by Theorem 2.1 that $\mathbb{P}$ satisfies the forcing theorem.

Lemma 6.2 suggests that one might try to use the class name $\dot{F}$ for the generic cofinal function $F\colon \kappa \to \mathrm{Ord}^M$ added by a non-pretame notion of class forcing $\mathbb{P}$ (by Lemma 2.2) to construct a forcing notion $\mathbb{Q}$ into which $\mathbb{P}$ densely embeds and such that there is a $\mathbb{Q}$-name τ for a subset of κ which has no nice $\mathbb{Q}$-name (the idea would be to obtain $\mathbb{Q}$ by adding the Boolean values of "$\dot{F}(\check{\alpha}) = \check{0}$" for $\alpha < \kappa$, which allow for the construction of a name τ for $\{\alpha < \kappa \mid F(\alpha) \neq 0\}$; in the case of $\mathrm{Col}(\omega, \mathrm{Ord})$, these Boolean values already exist). This approach would indeed work if $\kappa = \omega$, as we can construct such τ by [HKS, Lemma 8.7]. Since we however do not know whether names for the complements of (nice) names for subsets of arbitrary ordinals always exist (for more on this topic, consult [HKS, Section 8]), we will instead work with a name for an intersection of two nice names in the following, making use of [HKS, Lemma 8.5].

Theorem 6.5. *Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a countable transitive model of* KM*. Then a notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ is pretame if and only if it is densely nice.*

Proof. Suppose first that $\mathbb{P}$ is pretame. It is straightforward to check that whenever there is a dense embedding $\pi\colon \mathbb{P} \to \mathbb{Q}$ in $\mathcal{C}$ for some notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$, then $\mathbb{Q}$ is also pretame. Then by Example 6.4 (2), every such $\mathbb{Q}$ is nice.

Conversely, suppose that $\mathbb{P}$ is not pretame. Since $\mathbb{P}$ satisfies the forcing theorem over $\mathbb{M}$ (because every notion of class forcing does so over a model of KM – see either [Ant15, Lemma 15] or [HKL$^+$, Corollary 5.8]), we may, without loss of generality, assume that $\mathbb{P} = \langle P, \leq_{\mathbb{P}}\rangle$ is an M-complete Boolean algebra, and we may also assume that $P = \mathrm{Ord}^M$. We will extend $\mathbb{P}$ to a notion of class forcing $\mathbb{Q}$ for $\mathbb{M}$ which is not nice and so that $\mathbb{P}$ is a dense subforcing of $\mathbb{Q}$. By Lemma 2.2 there are a class name $\dot{F} \in \mathcal{C}^{\mathbb{P}}$, $\kappa \in \mathrm{Ord}$ and $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$\dot{F} : \check{\kappa} \to \mathrm{Ord}$ is cofinal". For the sake of simplicity, suppose that $p = \mathbb{1}_{\mathbb{P}}$.

For every $\alpha, \beta < \kappa$ and $p \in \mathbb{P}$, let

$$X_{p,\alpha,\beta} = \{\langle \gamma, \delta\rangle \in \mathrm{Ord}^M \times \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p\, [q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma} \wedge \dot{F}(\check{\beta}) = \check{\delta}]\},$$

and let

$$Y_{p,\alpha} = \{\gamma \in \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p\, [q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}]\}.$$

Claim 1. *For each $p \in \mathbb{P}$ there is $\alpha < \kappa$ such that for all $\beta < \kappa$, $X_{p,\alpha,\beta}$ is a proper class.*

Proof. Suppose the contrary. Then for every $\alpha < \kappa$ there exists $\beta_\alpha < \kappa$ such that $X_{p,\alpha,\beta_\alpha}$ is set-sized. In particular, this implies that for every $\alpha < \kappa$, $\{\gamma \in \mathrm{Ord}^M \mid \exists q \leq_{\mathbb{P}} p\, [q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}]\}$ is set-sized. But then p forces that the range of $\dot{F}$ is bounded in the ordinals, a contradiction. □

Let $C = \langle C_i \mid i \in \mathrm{Ord}^M\rangle \in \mathcal{C}$ be an enumeration of subclasses of $\mathrm{Ord}^M \times \mathrm{Ord}^M$ such that each C_i is of the form $X_{p,\alpha,\beta}$ for some $p \in \mathbb{P}$ and $\alpha, \beta < \kappa$ such that $X_{p,\alpha,\beta}$ is a proper class, and moreover each $X_{p,\alpha,\beta}$ which is a proper class appears unboundedly often in the enumeration C.

We will next perform a recursive construction to build two classes $D, E \in \mathcal{C}$, in a way that in particular each of $D \cap E$, $D \setminus E$ and $E \setminus D$ has a proper class sized intersection with $Y_{p,\alpha} = \{\gamma \mid \langle \gamma, \gamma\rangle \in X_{p,\alpha,\alpha}\}$ whenever $Y_{p,\alpha}$ is a proper class. The construction of the classes D, E will satisfy further properties which will be used in the proof of Claim 2 below.

Let $D_0 = D'_0 = E_0 = E'_0 = \emptyset$. Suppose that D_i, D'_i, E_i, E'_i have already been defined such that $D_i \cap D'_i = E_i \cap E'_i = D'_i \cap E'_i = \emptyset$ and $D_i \cup D'_i = E_i \cup E'_i$. We define $F_i = D_i \cup D'_i = E_i \cup E'_i$. Let $\langle \gamma_0, \delta_0\rangle, \langle \gamma_1, \delta_1\rangle, \langle \gamma_2, \delta_2\rangle$ be the lexicographically least pairs of ordinals in C_i such that each pair $\langle \gamma_k, \delta_k\rangle$ contains at least one ordinal not in $F_i \cup \{\gamma_j \mid j < k\} \cup \{\delta_j \mid j < k\}$, and γ_0, δ_0 additionally satisfy (if possible)

$$\gamma_0 \notin F_i \wedge \delta_0 \notin D'_i, \tag{1}$$

and γ_1, δ_1 satisfy in addition (if such exist)

$$\gamma_1 \notin F_i \cup \{\gamma_0, \delta_0\} \wedge \delta_1 \notin E'_i. \tag{2}$$

In the successor step, we will enlarge D_i, D'_i, E_i and E'_i to $D_{i+1}, D'_{i+1}, E_{i+1}$ and E'_{i+1} by putting distinct ordinals, which are not in F_i, into the sets $D_{i+1} \cap E_{i+1}$, $D_{i+1} \cap E'_{i+1}$ and $D'_{i+1} \cap E_{i+1}$. First, we put each ordinal in $\{\gamma_0, \delta_0\}$ which is not in F_i into $D_{i+1} \cap E'_{i+1}$. Next, we put all ordinals amongst $\{\gamma_1, \delta_1\}$ that are not in $F_i \cup \{\gamma_0, \delta_0\}$ into $D'_{i+1} \cap E_{i+1}$. Finally, we put every ordinal in $\{\gamma_2, \delta_2\}$ which is not yet in $F_i \cup \{\gamma_0, \gamma_1, \delta_0, \delta_1\}$ into $D_{i+1} \cap E_{i+1}$. Note that by construction, $D_{i+1} \cap D'_{i+1} = E_{i+1} \cap E'_{i+1} = D'_{i+1} \cap E'_{i+1} = \emptyset$ and $D_{i+1} \cup D'_{i+1} = E_{i+1} \cup E'_{i+1}$.

At limit stages, we take unions, e.g. if j is a limit ordinal, we let $D_j = \bigcup_{i<j} D_i$. Finally, let $D = \bigcup_{i\in\mathrm{Ord}^M} D_i \in \mathcal{C}$ and let $E = \bigcup_{i\in\mathrm{Ord}^M} E_i \in \mathcal{C}$.

Note that at each stage i such that $C_i = X_{p,\alpha,\alpha}$ for some $p \in \mathbb{P}$ and $\alpha \in \mathrm{Ord}^M$, each of the classes $D \cap E$, $D \setminus E$ and $E \setminus D$ obtains a new element from $Y_{p,\alpha}$. Since there are class many such stages, each of $D \cap E$, $D \setminus E$ and $E \setminus D$ has a proper class sized intersection with $Y_{p,\alpha}$ whenever $Y_{p,\alpha}$ is a proper class.

Let $a = \{\alpha < \kappa \mid \exists p \in \mathbb{P}\, [p \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \check{D}]\}$ and let $b = \{\alpha < \kappa \mid \exists p \in \mathbb{P}\, [p \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \check{E}]\}$. We extend $\mathbb{P}$ to a forcing notion $\mathbb{Q}$ by adding suprema for each of the classes

$$R_\alpha = \{p \in \mathbb{P} \mid p \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) \in \check{D}\} \text{ and}$$
$$S_\beta = \{p \in \mathbb{P} \mid p \Vdash_{\mathbb{P}} \dot{F}(\check{\beta}) \in \check{E}\}$$

for $\alpha \in a$ and $\beta \in b$, as described in Section 1. Let $p_\alpha = \sup_{\mathbb{Q}} R_\alpha$ and let $q_\beta = \sup_{\mathbb{Q}} S_\beta$ for $\alpha \in a$ resp. $\beta \in b$. Since $\mathbb{M}$ is a model of KM, $\mathbb{Q}$ satisfies the forcing theorem.

We will show that $\mathbb{Q}$ is not nice. Let $\dot{G}$ denote the canonical class name for the $\mathbb{Q}$-generic filter. Consider the $\mathbb{Q}$-names

$$\sigma = \{\langle \check{\alpha}, p_\alpha\rangle \mid \alpha \in a\} \text{ and } \tau = \{\langle \check{\alpha}, q_\alpha\rangle \mid \alpha \in b\}$$

for $\{\alpha < \kappa \mid \dot{F}^{\dot{G}}(\alpha) \in \check{D}\}$ and $\{\alpha < \kappa \mid \dot{F}^{\dot{G}}(\alpha) \in \check{E}\}$ respectively. It follows from [HKS, Lemma 8.4] that for every $\mathbb{Q}$-generic filter G there is a $\mathbb{Q}$-name μ such that $\mu^G = \sigma^G \cap \tau^G$. We claim that $M^{\mathbb{Q}}$ contains no nice name for $\sigma^G \cap \tau^G$. Suppose for a contradiction that there are $p \in \mathbb{Q}$ and a nice name $\nu \in M^{\mathbb{Q}}$ such that $p \Vdash_{\mathbb{Q}} \nu = \sigma \cap \tau$. By density of $\mathbb{P}$ in $\mathbb{Q}$, we may assume that $p \in \mathbb{P}$. Since ν is a nice name, it is of the form

$$\nu = \bigcup_{\alpha<\kappa} \{\check{\alpha}\} \times A_\alpha,$$

where each $A_\alpha \subseteq \mathbb{Q}$ is a set-sized antichain in M.

Let $\alpha < \kappa$ be as in Claim 1. We may assume that A_α only contains conditions which are compatible with p.

Claim 2. *For every $q \in A_\alpha$,*

$$Z_q = \{\gamma \in \mathrm{Ord}^M \mid \exists r \in \mathbb{P}\, [r \leq_{\mathbb{Q}} p, q \textit{ and } r \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}]\}$$

is a set in M.

Proof. We first consider $q \in A_\alpha \cap \mathbb{P}$. By assumption p and q are compatible, and since $\mathbb{P}$ is a Boolean algebra, $Z_q = Y_{p\wedge q,\alpha}$. Assume for a contradiction that $Y_{p\wedge q,\alpha}$ is a proper class. Then by our construction, $Y_{p\wedge q,\alpha} \setminus D$ is a proper class as well. Take $\gamma \in Y_{p\wedge q,\alpha} \setminus D$ and $r \leq_{\mathbb{P}} p \wedge q$ with $r \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$. Let G be $\mathbb{Q}$-generic with $r \in G$. Then $p, q \in G$ and so $\alpha \in \nu^G = \sigma^G \cap \tau^G$. On the other hand, since $\dot{F}^G(\alpha) = \gamma \notin D$, we have $p_\alpha \notin G$ and hence $\alpha \notin \sigma^G$. This is a contradiction.

Next, suppose that $q = p_\alpha$ and assume for a contradiction that Z_{p_α} is a proper class. Then $Y_{p,\alpha}$ is a proper class, so $Y_{p,\alpha} \cap (D \setminus E)$ is also a proper class. Now let $r \leq_{\mathbb{P}} p$ and $\gamma \in D \setminus E$ be such that $r \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$. Then $r \leq_{\mathbb{Q}} p_\alpha$ by the definition of p_α. If G is $\mathbb{Q}$-generic with $r \in G$, then $\alpha \in \nu^G$. Since $\gamma \notin E$, we have $\alpha \notin \tau^G$. This is a contradiction.

Next, suppose that $q = p_\beta \in A_\alpha$ for some $\beta \neq \alpha$. If there is some $\langle \gamma, \delta\rangle \in X_{p,\alpha,\beta}$ such that $\delta \in D$ but $\gamma \notin D \cap E$, then take $r \leq_{\mathbb{P}} p$ such that $r \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma} \wedge \dot{F}(\check{\beta}) = \check{\delta}$ and a $\mathbb{Q}$-generic filter containig r. Since $\delta \in D$ we have $p_\beta \in G$ and so $\alpha \in \nu^G$. On the other hand, $\dot{F}^G(\alpha) = \gamma \notin D \cap E$, so $\alpha \notin \sigma^G \cap \tau^G$. So there can be no such $\langle \gamma, \delta\rangle \in X_{p,\alpha,\beta}$. Hence for all $\langle \gamma, \delta\rangle \in X_{p,\alpha,\beta}$, if $\delta \in D$ then $\gamma \in D \cap E$. Suppose for a contradiction that Z_{p_β} is a proper class. Consider now the first stage i such that $X_{p,\alpha,\beta} = C_i$. Since $Y_{p,\alpha}$ is a proper class, there is $\langle \gamma, \delta\rangle \in X_{p,\alpha,\beta}$ such that $\gamma \notin F_i$. If there is such a pair which additionally satisfies that $\delta \notin D'_i$, then we are in case (1) in the recursive construction of D and E and so this would imply that γ ends up in $D \setminus E$ and $\delta \in D$. But we have already shown that this is impossible. So for every pair $\langle \gamma, \delta\rangle \in X_{p,\alpha,\beta}$ with $\gamma \notin F_i$ we have $\delta \in D'_i$. In particular, if $\delta \in D$ then $\gamma \in F_i$. But this implies that $Z_{p_\beta} \subseteq F_i$ is not a proper class, which is a contradiction.

The case $q = q_\alpha$ is analogous to the case $q = p_\alpha$. Finally, suppose that $q = q_\beta$ for some $\beta \neq \alpha$. As in the previous case $q = p_\beta$, we can conclude that for all $\langle \gamma, \delta\rangle \in X_{p,\alpha,\beta}$, if $\delta \in E$ then $\gamma \in D \cap E$. As above, we assume that Z_{q_β} is not in M and we let i be the least ordinal such that $C_i = X_{p,\alpha,\beta}$. After choosing γ_0, δ_0 in the recursive construction of D and E, there is still a pair $\langle\gamma_1, \delta_1\rangle$ such that $\gamma_1 \notin F_i^+ = F_i \cup \{\gamma_0, \delta_0\}$, since $Y_{p,\alpha}$ is a proper class. If possible, this pair is chosen such that $\delta_1 \notin E_i'$. But then γ_1 is put into $E \setminus D$ and δ_1 ends up in E. However, we have already argued that this cannot occur. But then for every such pair $\langle\gamma_1, \delta_1\rangle \in X_{p,\alpha,\beta}$ with $\gamma_1 \notin F_i^+$, we have $\delta_1 \in E_i'$, and so Z_{q_β} is contained in the set F_i^+, which is a contradiction. □

By Claim 2 and since $A_\alpha \in M$, we have that

$$B = \bigcup_{q \in A_\alpha} Z_q \in M.$$

Since $Y_{p,\alpha}$ is a proper class, so is $Y_{p,\alpha} \cap D \cap E$ by our construction, and hence there must be some $\gamma \in (Y_{p,\alpha} \cap D \cap E) \setminus B$. Let now $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}} \dot{F}(\check{\alpha}) = \check{\gamma}$ and take a $\mathbb{Q}$-generic filter G with $q \in G$. Then $\dot{F}^G(\alpha) = \gamma \in D \cap E$, so $\alpha \in \sigma^G \cap \tau^G$. Therefore there is some $r \in A_\alpha \cap G$. Take $s \in G$ with $s \leq_{\mathbb{Q}} q, r$. Then $s \Vdash_{\mathbb{Q}} \check{\gamma} = \dot{F}(\check{\alpha}) \in \check{B}$, contradicting the choice of γ. □

Theorem 6.6. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M. A notion of class forcing $\mathbb{P}$ for $\mathbb{M}$ which satisfies the forcing theorem satisfies the* Ord-cc *if and only if it is densely very nice.*

Proof. Suppose first that $\mathbb{P}$ satisfies the Ord-cc and $\mathbb{P}$ embeds densely into $\mathbb{Q}$. It is easy to see that then $\mathbb{Q}$ also satisfies the Ord-cc and so by Example 6.4, (3) it is very nice.

Conversely, suppose that $\mathbb{P}$ contains a class-sized antichain. We would like to extend $\mathbb{P}$ via a dense embedding to a partial order which is not very nice. Since $\mathbb{P}$ satisfies the forcing theorem, $\mathbb{P}$ has a Boolean M-completion. As we are only interested in a dense property, we may therefore assume that $\mathbb{P}$ is already an M-complete Boolean algebra.

By the proof of Lemma 4.4, we can find three disjoint subclasses of our given class-sized antichain, each of which contains a subclass which does not have a supremum in $\mathbb{P}$. Denote these subclasses without suprema by A, D and E, and let $B = A \cup D$ and $C = A \cup E$.

Claim 1. *At least one of* $\sup B$ *and* $\sup C$ *does not exist in* $\mathbb{P}$.

Proof. We show that if both $\sup B$ and $\sup C$ exist, then so does $\sup A$, contradicting our choice of A. Since $\mathbb{P}$ is an M-complete Boolean algebra, if $\sup B$ and $\sup C$ exist, then so does $p = \sup B \wedge \sup C$. We claim that p is already the supremum of A. It is clear that every element of A is below p. It remains to check that A is predense below p. Let $q \leq_{\mathbb{P}} p$. Since B is predense below q, there are $r \leq_{\mathbb{P}} q$ and $b \in B$ with $r \leq_{\mathbb{P}} b$. Since C is predense below r, there are $s \leq_{\mathbb{P}} r$ and $c \in C$ with $s \leq_{\mathbb{P}} c$. In particular, b and c are compatible. But they both belong to the antichain $B \cup C$, so $b = c \in B \cap C = A$. □

Let $\mathbb{Q}$ be the forcing notion obtained from $\mathbb{P}$ by adding $\sup B$ and $\sup C$. By Lemma 1.9, $\mathbb{Q}$ satisfies the forcing theorem. Moreover, it follows from the separativity of $\mathbb{P}$ that $\mathbb{Q}$ is separative. We show that $\mathbb{Q}$ is not very nice. Consider the $\mathbb{Q}$-name

$$\sigma = \{\langle\{\langle\check{0}, \sup B\rangle\}, \sup C\rangle\}.$$

By definition, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma \subseteq \check{2}$.

Claim 2. *There is no nice $\mathbb{Q}$-name τ such that $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \tau$.*

Proof. Suppose for a contradiction that $\tau = \{\check{0}\} \times A_0 \cup \{\check{1}\} \times A_1$, where $A_0, A_1 \in M$ are antichains of $\mathbb{Q}$, and $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \sigma = \tau$. Observe that $A_1 \subseteq \mathbb{P}$, since if for example $\sup B \in A_1$ and G is $\mathbb{Q}$-generic with $\sup B \in G$ and $\sup C \notin G$, then $1 \in \tau^G \setminus \sigma^G$. The same works for $\sup C$. Therefore $\sup A_1$ exists in $\mathbb{P}$. We claim that $\sup A_1$ is the supremum of A.

Firstly, we show that every element of A is below $\sup A_1$. Suppose for a contradiction that there is $a \in A$ with $a \nleq_{\mathbb{Q}} \sup A_1$. Then by separativity of $\mathbb{Q}$ there is $p \leq_{\mathbb{P}} a$ with $p \perp_{\mathbb{P}} \sup A_1$. In particular, p is incompatible with every element of A_1. Hence if G is a $\mathbb{Q}$-generic filter with $p \in G$ then $1 \in \sigma^G \setminus \tau^G$, contradicting our assumptions on σ and τ. Secondly, we check that A is predense below $\sup A_1$. Assume, towards a contradiction, that there is $p \leq_{\mathbb{P}} \sup A_1$ with $p \perp_{\mathbb{P}} a$ for each $a \in A$. Now A_1 is predense below p, so there exist $q \leq_{\mathbb{P}} p$ and $r \in A_1$ with $q \leq_{\mathbb{P}} r$. Again, this yields that for any $\mathbb{Q}$-generic filter G with $q \in G$, $1 \in \tau^G$ but $A \cap G = \emptyset$, so it is impossible that $\sup B$ and $\sup C$ are both in G. Hence $1 \notin \sigma^G$, contradicting our assumptions on σ and τ.

We have thus shown that $\sup A$ exists in $\mathbb{P}$, contradicting our choice of A. □

This proves that $\mathbb{Q}$ is not very nice. $\square$

The proof of Theorem 6.6 actually shows that every notion of forcing $\mathbb{P}$ which satisfies the forcing theorem but not the Ord-cc, can be densely embedded into a notion of class forcing which satisfies the forcing theorem and is nice but not very nice. To see this, it remains to check that the partial order $\mathbb{Q}$ constructed above is nice. This follows from the following more general result:

Lemma 6.7. *Suppose that $\mathbb{P}$ is a notion of class forcing which satisfies the forcing theorem. If $\mathbb{P}$ is nice and $\mathbb{Q}$ is obtained from $\mathbb{P}$ by adding the supremum of some subclass $A \in \mathcal{C}$ of $\mathbb{P}$, then $\mathbb{Q}$ is also nice.*

Proof. Let $\sigma \in M^{\mathbb{Q}}$ and $p \Vdash_{\mathbb{Q}} \sigma \subseteq \check{\gamma}$ for some $p \in \mathbb{Q}$ and $\gamma \in \mathrm{Ord}^M$. Let σ^+ denote the $\mathbb{P}$-name obtained from σ by replacing every ocurrence of $\sup A$ in $\mathrm{tc}(\sigma)$ by $\mathbb{1}_{\mathbb{P}}$, and let σ^- be defined recursively by $\sigma^- = \{\langle \tau^-, p\rangle \in \sigma \mid p \neq \sup A\}$. Let $q \leq_{\mathbb{Q}} p$. Without loss of generality, we can assume that $q \in \mathbb{P}$. If q is incompatible with every element of A, then $q \Vdash_{\mathbb{Q}} \sigma = \sigma^-$. But then there are $r \leq_{\mathbb{P}} q$ and a nice $\mathbb{P}$-name τ such that $r \Vdash_{\mathbb{P}} \sigma^- = \tau$ and so $r \Vdash_{\mathbb{Q}} \sigma = \tau$. If there is some $a \in A$ such that q is compatible with a, let $r \leq_{\mathbb{P}} q, a$. Then $r \Vdash_{\mathbb{Q}} \sigma = \sigma^+$ and so as in the previous case we can strengthen r to some s which witnesses that σ^+ has a nice $\mathbb{P}$-name. $\square$

Corollary 6.8. *Suppose that $\mathbb{M} = \langle M, \mathcal{C}\rangle$ is a countable transitive model of GB^- such that $\mathcal{C}$ contains a good well-order of M. Every notion of class forcing which satisfies the forcing theorem but not the* Ord-*cc is dense in a notion of class forcing which is nice but not very nice.* $\square$

7. Open Questions

Theorems 2.1 and 3.1 show that under the assumption that the ground model $\mathbb{M}$ has a hierarchy, pretameness implies the forcing theorem and is equivalent to the preservation of GB^-. It is therefore natural to ask whether this can already be shown in GB^-.

Question 7.1. *Is the assumption that $\mathbb{M}$ has a hierarchy necessary for Theorem 2.1, i.e. is there a model $\mathbb{M}$ of GB^- and a notion of class forcing for $\mathbb{M}$ which is pretame but does not satisfy the forcing theorem? Can pretameness be characterized by the preservation of GB^- in models without a hierarchy?*

Likewise, we ask the same question for the Ord-cc.

Question 7.2. *Is there a model $\mathbb{M} = \langle M, \mathcal{C}\rangle$ of GB^- and a notion of class forcing $\mathbb{P}$ which satisfies the* Ord-*cc but not the forcing theorem?*

In Section 5 we proved that in case there is a good well-order of the ground model, pretameness is equivalent to the EMP for partial orders which satisfy the forcing theorem. It is not known whether this can be generalized.

Question 7.3. *Assume $\mathbb{P}$ is a notion of forcing for $\mathbb{M}$ which does not satisfy the forcing theorem. Does this imply that the EMP fails for $\mathbb{P}$? Can pretameness be characterized by the EMP in the absence of a good well-order?*

In order to prove Theorem 6.5 we work in KM, since we need that the forcing theorem is preserved when adding infinitely many suprema to a given notion of class forcing that satisfies the forcing theorem. Thus the following question arises.

Question 7.4. *Can pretameness be characterized in terms of the existence of nice names for sets of ordinals in a theory that is substantially weaker than KM?*

Every proof of the failure of the forcing theorem known to the authors (see [HKL+, Section 7] and Theorem 2.6 of the present paper) uses the nonexistence of a first-order truth predicate in the ground model. This motivates the following question.

Question 7.5. *Assume $\mathbb{M} = \langle M, \mathcal{C}\rangle \models \mathsf{GB}^-$ and $\mathbb{N} = \langle M, \mathcal{C} \cup \{T\}\rangle$ where T is an $\mathbb{M}$-truth predicate. Does every notion of class forcing for $\mathbb{M}$ satisfy the forcing theorem over $\mathbb{N}$?*

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